# The relative Dehn method for coboundary expansion

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**Abstract**

We describe a new tool, which we call the *relative Dehn method*, for proving 1-dimensional coboundary expansion of coset complexes. By applying this method to links, we prove that certain "$B_3$-type coset complexes" constructed by O'Donnell and Pratt [OP22] are 1-cosystolic expanders over $\mathbb{F}_2$ and $\mathbb{F}_3$. Our method is suited to proving expansion over *specific* coefficient rings, whereas the pre-existing method due to Kaufman and Oppenheim [KO21], which we now term the *absolute Dehn method*, proves expansion over all rings (and even over nonabelian groups). Their method applies to the "$A_3$-type coset complexes" [KO18], but appears to be too strong for the "$B_3$-type complexes", which appear to *not* expand over certain coefficient groups. Our method requires using a computer to verify vanishing homology of certain "base case" complexes and then "lifting" using group-theoretic derivations.

# 1 Introduction

## 1.1 Background

**Defining simplicial complexes.** We begin by defining simplicial complexes.

**Definition 1.1.** A *(finite, pure) $d$-dimensional simplicial complex* $\mathfrak{X}$ is defined by a finite collection, denoted $\mathfrak{X}(d)$, of sets of size $d+1$. We term the elements in $\mathfrak{X}(d)$ *facets*.[1] We define the *vertices* of $\mathfrak{X}$ as the set of elements occurring in any facet:

$$V(\mathfrak{X}) \coloneqq \bigcup_{\sigma \in \mathfrak{X}(d)} \sigma.$$

Given a simplicial complex $\mathfrak{X}$, we define the set $\mathfrak{X}(i)$ of *$i$-faces* for $0 \le i \le d$ as size-$(i+1)$ sets contained in any facet:

$$\mathfrak{X}(i) \coloneqq \{\sigma \subseteq V(\mathfrak{X}) : |\sigma| = i+1 \wedge \exists \tau \in \mathfrak{X}(d) \text{ such that } \tau \supseteq \sigma\}.$$

We identify the 0-faces (elements of $\mathfrak{X}(0)$) with the vertices $V(\mathfrak{X})$.[2] The 1-faces (elements of $\mathfrak{X}(1)$) are unordered pairs of vertices, and we call them *edges*. The 2-faces (elements of $\mathfrak{X}(2)$) are unordered triples of vertices, and we call them *triangles*. ♢

**Definition 1.2.** A *weighted* simplicial complex is defined by a simplicial complex $\mathfrak{X}$ together with a fully supported probability distribution $\pi_d$ on the facets $\mathfrak{X}(d)$. $\pi_d$ induces a distribution $\pi_i$ on the $i$-faces $\mathfrak{X}(i)$; this is the distribution that samples a random facet $\boldsymbol{\sigma} \sim \pi_d$ and then sampling a uniformly random size-$(i+1)$ subset of $\boldsymbol{\sigma}$. ♢

Thus, simplicial complexes generalize graphs (indeed, a 1-dimensional simplicial complex is precisely a graph with no isolated vertices). The (weighted) graph corresponding to the vertices and edges of a complex $\mathfrak{X}$ is called the 1-*skeleton* of $\mathfrak{X}$.

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[1] We are only concerned with pure simplicial complexes in this paper, wherein all facets have the same size $(d+1)$. More generally, a simplicial complex is any downward-closed collection of subsets (called "faces"), wherein maximal sets ("facets") may have different sizes.

[2] Technically, this is an abuse of notation, since $\mathfrak{X}(0)$ actually equals the set of singleton sets of vertices, i.e., $\mathfrak{X}(0) = \{\{v\} : v \in V(\mathfrak{X})\}$).

**Introduction to high-dimensional expansion.** The concept of *high-dimensional expansion (HDX)* of simplicial complexes and related objects has played an important role in quite a number of exceptional recent breakthroughs: e.g., in the analysis of Markov chains [AOV21; ALOV24], in coding theory [EKZ24; DEL+22; PK22], in quantum complexity [ABN23], property testing [DK17; DDL24; BM24; BLM24], inapproximability [BMVY25], and in constructions of vertex expander graphs [HLM+25a; HLM+25b].Very informally, a $d$-dimensional simplicial complex is an "HDX" if it is "sparse" and "well-connected" in a spectral or topological sense.

In the case $d = 1$, simplicial complexes are graphs, and "HDX-ness" is simply the classical notion of *expander graphs*. Recall that for graphs, there are two main definitions for expansion, edge expansion and spectral expansion, between which Cheeger's inequality [Alo86] gives a bidirectional relation.

Unfortunately, it turns out that once $d > 1$, even *defining* HDX becomes subtle, because there is a glut of inequivalent notions of expansion. Indeed, applications of HDXs often seem to use bespoke notions of expansion suited for particular situations. As an example, the aforementioned recent breakthroughs in agreement testing [BLM24; DDL24] and low-soundness PCPs [BMVY25] needed HDXs with both good "spectral expansion" and a very strong variant of "coboundary expansion" — for which currently only a single family of simplicial complexes, the "$\widetilde{C}_d$-type affine building quotients" due to [CL25], fits the bill.

**Our setting: 1-coboundary expansion.** The 0-*coboundary expansion* of a complex is the edge expansion of its 1-skeleton in the usual graph-theoretic sense. In this work, we study one particular notion of high-dimensional expansion called "1-*coboundary expansion*", which is a "higher-dimensional" analogue of 0-coboundary expansion. 1-coboundary expansion is defined with respect to a "coefficient group" $\Gamma$ (which may even be nonabelian; a complex might be a good 1-coboundary expander over one $\Gamma$ but not another $\Gamma'$.

Informally, 1-coboundary expansion can be viewed as a property testing guarantee for certain natural constraint satisfaction problems (CSPs) associated to simplicial complexes, where the alphabet is $\Gamma$, variables correspond to vertices, and constraints correspond to edges. We do not define 1-coboundary expansion formally in this introduction; instead, we describe a condition which implies 1-coboundary expansion in the specific setting of *symmetric* complexes (with the uniform weight distribution), which is all that we will need for the purposes of this paper. (See §2.3.1 below for the definition of coboundary expansion for general complexes.)

It is a well-known fact in graph theory that an edge-symmetric graph with diameter $R_0$ has edge-expansion (a.k.a. 0-coboundary expansion) at least $1/(2R_0)$ (see, e.g., [Chu96]). In a (symmetric) complex of dimension at least 2, 1-dimensional coboundary expansion over $\Gamma$ is implied by a "higher-dimensional" isoperimetric property, which we call *tautness*. We say a complex $\mathfrak{X}$ is $(R_0, R_1)$-*(homologically) taut over* $\Gamma$ if every "loop" of length at most $R_0$ in $\mathfrak{X}$ has "area at most $R_1$ over $\Gamma$" (we shall explain these terms in more detail momentarily). By the so-called *cones method* [Gro10; KO21], a symmetric complex $\mathfrak{X}$ with diameter $R_0$ which is $(2R_0 + 1, R_1)$-homologically taut over $\Gamma$ has 1-coboundary expansion at least $1/R_1$ over $\Gamma$ (see Definition 2.35 and Theorem 2.36 below).

Forgetting about $\Gamma$ momentarily, very roughly, a 2-dimensional simplicial complex can be visualized as a surface in 3-dimensional space, and the area of a loop $L$ on the surface is the smallest area of any subsurface whose boundary is $L$. To be more formal, a *proper loop* of *length* $L$ in $\mathfrak{X}$ is simply a sequence of vertices $(u_0, u_1, \ldots, u_{L-1}, u_L = u_0)$ such that each successive pair of vertices is an edge, i.e., $\{u_\ell, u_{\ell+1}\} \in \mathfrak{X}(1)$. The *area* of a loop $L$ over a ring $\Gamma$ is a quantity denoted $\triangle^{\Gamma}_{\mathrm{H}}(L) \in \mathbb{N} \cup \{\infty\}$. $(R_0, R_1)$-homological tautness of $\mathfrak{X}$ over $\Gamma$ posits that every proper loop $L$ of length in $\mathfrak{X}$ at most $R_0$ satisfies $\triangle^{\Gamma}_{\mathrm{H}}(L) \le R_1$.

In the case $\Gamma = \mathbb{Z}_2$, it is not too hard to formally define area: Given a loop $L$, a *filling* of $L$ is a set of triangles $T \subseteq \mathfrak{X}(2)$ whose *boundary* is $L$, where the *boundary* of a set of triangles $T$ is the set of edges incident to an odd number of triangles in $T$. $\triangle\mathbb{Z}_2(L)$ equals size of the smallest *filling* of $L$ if one exists, and $\infty$ if none exists. To define area over groups aside from $\mathbb{Z}_2$, we need a bit more setup, in particular notions like oriented triangles and antisymmetric functions, but the definition is intuitively the same.

**Connectedness.** We say that a simplicial complex $\mathfrak{X}$ is 0-*connected* if its 1-skeleton is connected in the usual graph-theoretic sense. This is equivalent to having finite diameter, and to having positive 0-coboundary expansion (i.e., edge expansion). Analogously, positive 1-coboundary expansion over $\Gamma$ is equivalent to a notion called *homological* 1-*connectedness over* $\Gamma$. Homological 1-connectedness over $\Gamma$ is also equivalent to the "absence of holes", in the sense that every loop has finite area over $\Gamma$.

**A weaker notion: 1-cosystolic expansion.** At times, it is convenient to also study a weaker variant of 1-coboundary expansion over $\Gamma$, called 1-*cosystolic expansion*. Morally, 1-cosystolic expansion is equivalent to 1-coboundary expansion *without* the "global" property of homological 1-connectedness; hence, "1-cosystolic expansion over $\Gamma$ + homological 1-connectedness over $\Gamma$ = 1-coboundary expansion over $\Gamma$". (For a 0-dimensional analogy: Suppose you have a graph and you want to show that it is an edge expander, but you do not know how to show that it is connected. Therefore, you settle for showing that each connected component of the graph is an edge expander with many vertices. This is precisely the notion of 0-cosystolic expansion.) The main advantage of studying cosystolic expansion, as opposed to coboundary expansion, is that it can be established via *local-to-global arguments*. These arguments are well-known at this point in the HDX literature; see, e.g., [EK16; DD24a]. At the same time, cosystolic expansion (even over $\mathbb{Z}_2$) is still powerful enough for interesting applications, including topological expansion [Gro10], locally testable and quantum LDPC codes [EKZ24; DEL+22; PK22], and sum-of-squares lower bounds [DFHT21; HL22].

We remark that we currently (to the best of our knowledge) do not even know how to prove 2-dimensional cosystolic or coboundary expansion for any (sparse) explicit complexes, and only know a handful of examples of complexes satisfying these conditions in 1 dimension. This motivates the current study, wherein we give new methods for proving and examples of complexes satisfying these conditions.

**Coset complexes.** Coset complexes are a well-studied way to build simplicial complexes from algebraic structures; they were first studied by Lannér [Lan50].

For a finite group $G$ and a finite set $\Lambda$, a $\Lambda$-*indexed subgroup family* $\mathcal{H}$ is a collection of subgroups $(H_\lambda < G)_{\lambda\in\Lambda}$.

**Definition 1.3.** Let $G$ be a finite group and $\mathcal{H} = (H_\lambda < G)_{\lambda\in\Lambda}$ a $\Lambda$-indexed subgroup family. The *coset complex* $\mathfrak{CC}(G;\mathcal{H})$ is the following $(|\Lambda|-1)$-dimensional simplicial complex:

- The vertices are partitioned into $|\Lambda|$ *types*, and the type-$\lambda$ vertices (for $\lambda \in \Lambda$) are elements of $G/H_\lambda$, i.e., (left) cosets of $H_\lambda$ in $G$.
- A tuple of cosets $(C_\lambda)_{\lambda\in\Lambda}$, where $C_\lambda$ has type $\lambda$, forms a facet iff the mutual intersection of cosets $\bigcap_{\lambda\in\Lambda} C_\lambda$ is nonempty, or equivalently there exists $g \in G$ such that $C_\lambda = gH_\lambda$ for every $\lambda \in \Lambda$.
- The distribution on facets $\pi_d$ is uniform.

It is not hard to check that a set of $i+1$ cosets $C_0,\dots,C_i$ of types $\lambda_0,\dots,\lambda_i$, respectively ($\lambda_0,\dots,\lambda_i$ all distinct), forms a $i$-face iff $\bigcap_{\ell=0}^{i} C_\ell \neq \emptyset$, or equivalently there exists $g \in G$ such that $C_\ell = gH_{\lambda_\ell}$ for every $\ell \in [i]$. $\Diamond$

Coset complexes have several nice properties. Firstly, they are *partite*: Facets have exactly one each vertex of each type. Secondly, they are *highly symmetric*: In particular, they satisfy a useful technical condition called "admitting a group acting transitively on facets" (in particular, this group is $G$). Thirdly, topological properties of a complex (like its connectivity and expansion) are intimately related to the structure of the group $G$ vis-à-vis its designated subgroups. For instance, we make the following definition:

**Definition 1.4.** Let $\bigcup\mathcal{H}$ denote the union of the subgroups $H_\lambda$ (regarded as a subset of $G$). A *word* of *length* $\ell$ over $\mathcal{H}$ is a sequence of elements, denoted $\langle g_0\rangle\cdots\langle g_{\ell-1}\rangle$, with each $g_i \in \bigcup\mathcal{H}$. (Here, the notation $\langle\cdot\rangle$ is used to emphasize that the length-1 word $\langle x\rangle$ is formally different than the subgroup element $x$ itself.) The inverse of a word $w = \langle g_0\rangle\cdots\langle g_{\ell-1}\rangle$ is the word $w^{-1} := \langle g_{\ell-1}^{-1}\rangle\cdots\langle g_0^{-1}\rangle$. The concatenation of two words $w = \langle g_0\rangle\cdots\langle g_{\ell-1}\rangle$ and $w' = \langle g'_0\rangle\cdots\langle g'_{\ell'-1}\rangle$ is the word $ww' := \langle g_0\rangle\cdots\langle g_{\ell-1}\rangle\langle g'_0\rangle\cdots\langle g'_{\ell'-1}\rangle$. $\Diamond$

**Definition 1.5.** The *evaluation* of a word $w = \langle g_0\rangle\cdots\langle g_{\ell-1}\rangle$, denoted $\mathsf{eval}(w)$, is the ordered product $g_0\cdots g_{\ell-1} \in G$. We have $\mathsf{eval}(w^{-1}) = (\mathsf{eval}(w))^{-1}$ and $\mathsf{eval}(ww') = \mathsf{eval}(w)\mathsf{eval}(w')$. $\Diamond$

**Proposition 1.6** (e.g., [HS24; KO21])**.** *Let $G$ be a finite group and $\mathcal{H} = (H_\lambda < G)_{\lambda\in\Lambda}$ an indexed subgroup family. Then:*

- Qualitative bound: *$\mathfrak{CC}(G;\mathcal{H})$ is 0-connected iff every element in $G$ can be written as the evaluation of some (finite-length) word over $\mathcal{H}$.*
- Quantitative bound: *If every element in $G$ can be written as the evaluation of a length-($\leq R_0$) word over $\mathcal{H}$. then the 1-skeleton of $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$.*

As observed by [KO21], this quantitative bound can already be combined with the well-known fact [BS92; Chu96] that a symmetric graph with diameter $R_0$ has edge expansion (at least) $\frac{1}{2R_0}$ to get 0-coboundary expansion (a.k.a. edge expansion) bounds for coset complexes. As we shall see in the next subsection, algebraic conditions also imply tautness and therefore 1-coboundary expansion in coset complexes.

## 1.2 Main theorem: The relative Dehn method

The main conceptual contribution of our work is introducing a new tool, which we term the *relative Dehn method*, for proving coboundary expansion of coset complexes. This method builds on a prior tool due to [KO21] which, by contrast, we term the *absolute Dehn method*. (See also Appendix C of this paper, and Theorem C.23 therein, for a restatement and reproof of the absolute Dehn method with coefficients in arbitrary groups and with mildly improved quantitative parameters.)

One important fact about the absolute Dehn method [KO21] is that it is *agnostic* to the coefficient group $\Gamma$. That is, when the method applies to a complex $\mathfrak{X}$, it implies homological 1-connectedness and coboundary expansion of $\mathfrak{X}$ over every $\Gamma$.[3] In contrast, our relative Dehn method is *aware* of the coefficient group $\Gamma$: It can be used to prove homological 1-connectedness and coboundary expansion of $\mathfrak{X}$ over $\Gamma$ even when $\mathfrak{X}$ is *not* homologically 1-connected over other groups $\Gamma'$. Hence, our relative Dehn method can handle some complexes which the absolute Dehn method of [KO21] cannot. One such example, which is the primary motivation for and application of our work, is the $B_3$-type coset complex of [OP22], which we describe in the following subsection.

Just as Proposition 1.6 above relates 0-connectedness and edge expansion with simple group-theoretic properties, the absolute and relative Dehn methods relate 1-*connectedness* and 1-*coboundary expansion* with group-theoretic properties. To state these conditions, we need some additional algebraic definitions. In this subsection, we fix a group $G$, an index set $\Lambda$, and an indexed subgroup family $\mathcal{H} = (H_\lambda < G)_{\lambda\in\Lambda}$.

**Definition 1.7.** We define an equivalence relation $\sim$ on words over $\mathcal{H}$ by specifying that $v \sim u$ if $v$ is a cyclic shift of either $u$ or $u^{-1}$, where (as expected) $u^{-1}$ denotes the word formed by reversing $u$ and replacing each symbol with the symbol for its inverse in $G$. (The semantics of $v \sim u$ is "$v \equiv \mathbb{1}$ iff $u \equiv \mathbb{1}$".) $\Diamond$

**Definition 1.8** (Subgroup relators)**.** A word $w = \langle x_0\rangle \cdots \langle x_{\ell-1}\rangle$ over $\mathcal{H}$ is a *relator* if its evaluation is $\mathbb{1}$ in $G$. We let $\mathcal{R}_\ell$ denote the set of all relators of length at most $\ell$. An *equation* over $\mathcal{H}$ is a string "$y \equiv z$", where $y, z$ are words over $\mathcal{H}$ and $yz^{-1}$ is a relator; we identify the equation with this relator. $\Diamond$

**Definition 1.9** (Derivations)**.** A relator $x$ is *derived from* a relator $y$ *via* relator $r$ if (for some words $p, q, u, v$ over $\mathcal{H}$),

$$x \sim p \circ u \circ q, \quad z \sim p \circ v \circ q, \quad \text{“}u \equiv v\text{”} \sim r. \tag{1.10}$$

In other words, up to equivalences, $x$ can be obtained from $y$ by substitution of an equation equivalent to $r$. Note that it is equivalent to reverse the derivation, i.e., to say that $y$ is derived from $x$ via $r$. $\Diamond$

**Definition 1.11** (Derivation length)**.** Let $w$ be a relator over $\mathcal{H}$ and $\mathcal{R}$ a set of relators. We write $\operatorname{area}(w; \mathcal{R})$ for the least number of derivation steps, via relators from $\mathcal{R}$, that it takes to reduce $w$ to the word $\mathbb{1}$. (Or, we write $\operatorname{area}(w; \mathcal{R}) = \infty$ if this is not possible.) If $w \sim$ "$x \equiv y$", we also write this as $\operatorname{area}(x \equiv y; \mathcal{R})$, and refer to it as the least number of steps required to derive "$x \equiv y$" via $\mathcal{R}$. $\Diamond$

This function $\operatorname{area}(w; \mathcal{R})$ is essentially the same as, though not quantitatively identical to, the *Dehn function* which is a widely studied measure of complexity in combinatorial group theory (see, e.g., [Bri02; Ril17] and Remarks 3.23 and 3.24 below).

The group-theoretic hypothesis for our relative Dehn method, and also for the absolute Dehn method of [KO21], concerns bounds on $\operatorname{area}(w; \mathcal{R})$ for specific sets of relators $\mathcal{R}$. For our relative Dehn method, given a commutative ring $\Gamma$ and parameters $\ell, t \in \mathbb{N}$, we define a certain set of relators, called the *$t$-fillable relators over* $\Gamma$ and denoted $\mathcal{R}_{\ell,t}(\triangle^\Gamma_{\mathrm{H}}) \subseteq \mathcal{R}_\ell$ (for a given length $\ell$). These are relations which have "area" at most $t$ in a certain sense. Formally defining $t$-fillability requires a bit more setup, which we defer to Definition 3.20 below, but we can now state our main theorem modulo this definition:

[3][KO21] does not explicitly claim expansion over every $\Gamma$, only over $\mathbb{Z}_2$, but their proof bounds the "spherical filling number" using the Dehn function, and in turn, their proof of coboundary expansion from bounds on the spherical filling number works over every $\Gamma$. See Appendix C.

**Theorem 1.12** (Relative Dehn method)**.** *Let $\Gamma$ be any commutative ring and consider the $d$-dimensional coset complex $\mathfrak{CC}(G;\mathcal{H})$. Suppose that there exist $R_0, \ell, t, \delta \in \mathbb{N}$ such that:*

1. *every element in $G$ can be written as the evaluation of a length-($\leq R_0$) word over $\mathcal{H}$.*

2. *every $w \in \mathcal{R}_{2R_0+1}$ satisfies $\operatorname{area}(w; \mathcal{R}_{\ell,t}(\triangle_{\mathrm{H}}^{\Gamma})) \leq \delta$.*

*Then $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$ and is $(2R_0+1, O((t+\ell)\delta))$-homologically taut over $\Gamma$, and therefore has 1-coboundary expansion at least $\Omega(\frac{1}{(t+\ell)\delta})$ over $\Gamma$.*

In contrast, the *absolute Dehn method* of Kaufman and Oppenheim [KO21] uses a different set of relators $\mathcal{R}_\ell^{\mathrm{common}}$:

**Definition 1.13.** $\mathcal{R}_\ell^{\mathrm{common}}$ is the set of relators $\langle g_0\rangle \cdots \langle g_{\ell'-1}\rangle$ ($\ell' \leq \ell$) such that every $g_i \in H_\lambda$ for a single common subgroup $\lambda \in \Lambda$. ◊

Note that $\mathcal{R}_\ell^{\mathrm{common}}$ is not defined with respect to $\Gamma$. Once the proper definition of fillability is given, it will be easy to check that $\mathcal{R}_\ell^{\mathrm{common}} \subseteq \mathcal{R}_{\ell,0}(\triangle_{\mathrm{H}}^{\Gamma})$ (for every $\ell \in \mathbb{N}$ and commutative ring $\Gamma$). The absolute Dehn method hypothesizes a bound on $\operatorname{area}(w; \mathcal{R}_\ell^{\mathrm{common}})$ for every $w \in \mathcal{R}_{2R_0+1}$ and concludes coboundary expansion bounds even for $\Gamma$ being a nonabelian group. Hence, the absolute Dehn method has a strictly stronger conclusion at the cost of a strictly stronger hypothesis.

**Remark 1.14.** There is a stronger notion of 1-connectedness, called *simple connectedness*; if a complex $\mathfrak{X}$ is simply connected, then it is homologically 1-connected over every coefficient group $\Gamma$ (including nonabelian groups). Though we will not need its definition in this paper, we remark that simple connectedness may be regarded as a "homotopy" variant of (homological) 1-connectedness. A corollary of the absolute Dehn method is that if $\operatorname{area}(w; \mathcal{R}_\ell^{\mathrm{common}})$ is *finite* for every $w \in \mathcal{R}_{2R_0+1}$, then $\mathfrak{CC}(G;\mathcal{H})$ is simply connected. This "qualitative" implication predates the absolute Dehn method of [KO21] itself; it has been known at least since the work of Lannér [Lan50] (see also [Bun52; Gar79; AH93]). ◊

Since $\mathcal{R}_\ell^{\mathrm{common}} \subseteq \mathcal{R}_{\ell,0}(\triangle_{\mathrm{H}}^{\Gamma})$, the relative Dehn method quantitatively recovers the absolute Dehn method over all commutative rings $\Gamma$.[4] (That is, if all relators of length at most $2R_0+1$ are derivable from $\mathcal{R}_\ell^{\mathrm{common}}$ in at most $\delta$ steps, then they are certainly derivable from $\mathcal{R}_{\ell,0}(\triangle_{\mathrm{H}}^{\Gamma})$ in at most $\delta$ steps for every $\Gamma$.) However, we note that the absolute Dehn method also applies when $\Gamma$ is not a commutative ring.

We remark that the technical tools involved in the proofs of the two methods do not seem very different. Our main contribution is that the relative Dehn method is a viable and conceptually distinct route for proving coboundary expansion for interesting complexes (such as the "$B_3$-type coset complex" of [OP22], see the following subsection) where the absolute Dehn method provably fails.

We now give some intuition for the relative Dehn method, and for the related notion of fillability. There is an easy-to-describe "correspondence" between loops in a coset complex $\mathfrak{CC}(G;\mathcal{H})$ and words over $\mathcal{H}$.[5] In particular, $\mathcal{R}_{\ell,t}(\triangle_{\mathrm{H}}^{\Gamma})$ is precisely the set of relators of length at most $\ell$ for which the corresponding loops have area at most $t$ over $\Gamma$. Thus, by the cones method [Gro10; KO21], to prove coboundary expansion $\Omega(1/R_1)$ in a coset complex, it suffices to show that $\mathcal{R}_{2R_0+1} \subseteq \mathcal{R}_{2R_0+1,R_1}(\triangle_{\mathrm{H}}^{\Gamma})$, i.e., all relations of length at most $2R_0+1$ have area at most $R_1$ over $\Gamma$. The relative Dehn method lets us *bootstrap* and get area bounds for all relations of a longer length $(2R_0+1)$ by algebraically deriving all such relations from fillable relations of a shorter length $(\ell)$.

To apply the relative Dehn method, one essentially needs to complete two tasks:

1. Establish that there are "useful" relators in the set $\mathcal{R}_{\ell,t}(\triangle_{\mathrm{H}}^{\Gamma})$.

2. Algebraically derive all relators of length at most $2R_0+1$ from $\mathcal{R}_{\ell,t}(\triangle_{\mathrm{H}}^{\Gamma})$.

The second task is a completely group-theoretic problem. In the $A_3$-type case, it was solved by Kaufman and Oppenheim for the $A_3$-type case in [KO21]; in the $B_3$-type case, it was solved in an earlier version of this

[4] See Appendix C, where we give a mildly quantitatively improved reproof of the absolute Dehn method which closely mirrors our proof of the relative Dehn method.

[5] We put correspondence in quotation marks, because it is not actually bijective: one loop corresponds to many relators and one relator corresponds to many loops. However, the correspondence is bijective on equivalence classes modulo some easy-to-define equivalence relations.

work. This solution was quite long and mechanical, and it is omitted from the current manuscript because it was subsequently formalized in [WBCS25].

For the first task, given that we are applying the relative Dehn method and not the absolute Dehn method, we will typically be in a situation where the relators in $\mathcal{R}_\ell^{\text{common}}$ are *not* enough to derive all of $\mathcal{R}_{2R_0+1}$. For the $B_3$-type case described in the following subsection, we get additional fillable relations via "lifting" from a small, fixed-size base complex, for which we check fillability using a computer.

## 1.3 Application: $B_3$-type coset complexes

The motivating application for our relative Dehn method (Theorem 1.12) is a family of coset complexes, called *Chevalley* complexes [KO18; OP22], and more specifically, the "$B_3$-type coset complex" of [OP22]. We defer defining these complexes formally until §4, but we now give some intuition for them and describe our result for them.

**Remark 1.15.** The original motivation for this work was to understand whether the $B_3$-type complex itself was a cosystolic expander over all (even nonabelian) groups $\Gamma$. This is one of the properties required of simplicial complexes for downstream applications in agreement testing and PCPs [BM24; DD24b; DDL24; BLM24; BMVY25]. Ultimately, our investigations showed that the $B_3$-type complex appears *not* to have this property, unlike the $A_3$-type complex, which was analyzed in [KO21] using the absolute Dehn method. We developed the relative Dehn method in order to still prove some cosystolic expansion bounds for the $B_3$-type, even if not as strong as would be needed for these downstream applications.

While we frame the current work mostly around this new method, a substantial reason we sought to study the $B_3$-type complex was to diversify the landscape of cosystolic-expanding complexes. Very roughly, [DDL24; BLM24] had to study certain "$C$-type" building complexes [CL25], instead of the existing "$A$-type" ones, in order to achieve requisite properties for downstream applications. We hoped that studying the $B$- (and $C$-)type coset complexes might therefore prove fruitful towards finding additional constructions of agreement testers and PCPs. This point may now be moot since requisite properties for the $A$-type coset complexes were subsequently established by Kaufman, Oppenheim, and Weinberger [KOW25]. $\Diamond$

Recall that to instantiate coset complexes, one needs groups admitting indexed families of subgroups. One natural choice turns out to be the *Chevalley groups*, which are matrix groups over finite fields. These groups have a lot of structure which is useful for building coset complexes and analyzing their HDX properties, including: they have natural and highly structured subgroups; they can be described succinctly via generators and relations (i.e., via presentations); and their spectral expansion has an elementary analysis [HS24; OP22]. The "$A_d$-type" Chevalley group used in [KO18] essentially corresponds to $(d+1)\times(d+1)$ determinant-1 matrices over a finite field (i.e., $\mathsf{SL}_{d+1}(\mathbb{F}_q)$) while the "$B_d$-type" Chevalley group used in [OP22], which we study in this paper, essentially corresponds to $(2d+1)\times(2d+1)$ determinant-1 "orthogonal" matrices over a finite field (i.e., $\mathsf{SO}_{2d+1}(\mathbb{F}_q)$). See [HS24] for an excellent overview of properties of coset complexes and the specific instantiation with $\mathsf{SL}_{d+1}(\mathbb{F}_q)$.

**Remark 1.16.** We emphasize that we now typically have two finite fields to keep in mind: (1) the field used to define the Chevalley group, and (2) the field over which we want to calculate coboundary expansion of the corresponding complex. These fields may have different sizes. $\Diamond$

Now, we recall some more details from [OP22] of the simplicial complex in question:

**Definition 1.17** (The "$B_3$-type coset complex")**.** Let $\mathbb{F}_q$ be a field of order $q$ and odd characteristic.[6] Let $\widetilde{\mathbb{F}}$ be an extension of degree $m \geq 6$, so $|\widetilde{\mathbb{F}}| = n \coloneqq q^m$. Let $B_3(\widetilde{\mathbb{F}})$ denote the universal $B_3$-type Chevalley group[7] over $\widetilde{\mathbb{F}}$, of cardinality $N \coloneqq n^9(n^6-1)(n^4-1)(n^2-1) \sim n^{21}$. Let $H_\alpha$, $H_\beta$, $H_\psi$, $H_\omega \subseteq B_3(\widetilde{\mathbb{F}})$ be the four subgroups defined in [OP22] (and in Definition 4.23 in this paper) of cardinality $q^{20}$, $q^{20}$, $q^{31}$, $q^{31}$ (respectively), and let $\mathfrak{B}_q^3(m)$ be the associated 3-dimensional coset complex, acted upon by $B_3(\widetilde{\mathbb{F}})$ in a tetrahedron-transitive way. This complex is on $V \coloneqq (2q^{-20} + 2q^{-31})N \sim 2q^{21m-20}$ vertices, with each vertex in $\Theta(q^{31})$ tetrahedra. $\Diamond$

[6] In [OP22], the "base field" was always described as having prime order $p$. But at no point was this ever required; inspection of the paper shows that everything is just the same (after replacing "$p$" by "$q$") for non-prime-order base fields $\mathbb{F}_q$ (of characteristic not 2). Also, though that paper's statements required characteristic at least 5, they noted that characteristic 3 was also fine for all types other than "$G_2$".

[7] $B_3(\widetilde{\mathbb{F}})$ is also known as $\Omega_7(\widetilde{\mathbb{F}})$, the commutator subgroup of $\mathsf{O}_7(\widetilde{\mathbb{F}})$, the group of $7\times 7$ orthogonal matrices over $\widetilde{\mathbb{F}}$.

**Remark 1.18.** The *smallest* possible expanding instantiation of the above complex ($q = 19$, $m = 6$) has $V > 2^{450}$. (Similar astronomical numbers hold for all known HDX families.) This illustrates the importance of *strong* explicitness; we can work with that HDX implicitly, computing incidences in time "poly(450)". $\Diamond$

By applying our relative Dehn method, we show that the $B_3$-type simplicial complexes are cosystolic expanders:

**Theorem 1.19** (Main application). *If $q$ is a sufficiently large power of 5, the 2-dimensional simplicial complexes $\widehat{\mathfrak{B}}^3_q(m)$ have 1-cosystolic expansion at least $(\epsilon_0, \mu_0)$ over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, where $\epsilon_0, \mu_0 > 0$ are universal constants.*

(See Definition 2.27 below for the formal definition of cosystolic expansion which we use.)

Just as in [KO21] for the $A_3$-type case, by well-known tools [Gro10; DKW18], Theorem 1.19 implies *topological expansion* of the complexes $\mathfrak{B}^3_q(m)$: If the graph-with-triangles $\widehat{\mathfrak{B}}^3_q(m)$ is (continuously) drawn in the plane, no matter how curvily, there will be a point in $\mathbb{R}^2$ contained in a constant fraction of its triangles.

Kaufman and Oppenheim [KO21] proved similar theorems for the family of "$A_3$-type coset complexes over $\mathbb{F}_q$" they had earlier constructed [KO18] using their *absolute Dehn method*, and which we denote $(\mathfrak{A}^3_q(m))_{m\geq 4}$. (Because of the power of the absolute Dehn method, their theorems held for $q$ being any sufficiently large prime and for homology over any finite field, or indeed, any even nonabelian group. We emphasize, again, that the absolute Dehn method cannot apply to the $B_3$-type complexes.)

**Remark 1.20.** Let us give a small bit of flavor of the work of [KO21]. The very first (of many) interesting group-theoretic tasks Kaufman and Oppenheim needed to solve, in order to prove cosystolic expansion of $\mathfrak{A}^3_q(m)$, was the following:

> Fix a prime $p$, and consider the group with generators $\alpha$, $\beta$, $\gamma$, $\lambda$, and $\mu$, each of order $p$. Assume that $\alpha$ and $\gamma$ commute with all other generators except $\beta$, that $\alpha\beta\alpha^{-1}\beta^{-1} = \lambda$, and that $\beta\gamma\beta^{-1}\gamma^{-1} = \mu$. Must $\lambda$ and $\mu$ commute, i.e., $\lambda\mu\lambda^{-1}\mu^{-1} = \mathbb{1}$?

Luckily for them, this was proven for $p = 3, 5, 7, 11, 13$ in [Kir78; Eve78] and for all odd $p$ in [BD01];[8] indeed, extending the [BD01] proof to generalized settings was the main component of their group-theoretic work. For our tackling of the $B_3$ case, there is a highly analogous but more complex problem that — if it could be proven — would likely lead to a proof of coboundary expansion over all $\Gamma$ using the absolute Dehn method. Unfortunately, it appears this analogue is false, and therefore the absolute Dehn method cannot be used![9] See Appendix A for a simple version of the [BD01] proof and more discussion. On the other hand, despite being unable to show the $B_3$ analogue of "$\lambda\mu\lambda^{-1}\mu^{-1} = \mathbb{1}$", we were able to use computer-assistance to verify that when $p = 5$, the "corresponding loop" is "homologically" fillable. This is what led us to develop the relative Dehn method, unlocking the ability to combine computer-discovered fillings with human-discovered group-theoretic proofs. $\Diamond$

Beyond our new relative Dehn method, several other ingredients go into the proof of Theorem 1.19:

1. Standard tools also used in [KO21] in the $A_3$ case, namely, local-to-global theorems for cosystolic expansion [KKL16; EK16; DD24a] and the cones method for relating coboundary expansion with homology fillings [Gro10; LMM16; KM19; KO21],
2. Bounds on the *spectral* expansion of the $B_3$-type coset complexes from [OP22] (analogous to results from [KO18] used by [KO21] in the $A_3$ case),
3. Computer calculations of "Betti numbers" of (links in) the $B_3$-type coset complexes over various finite fields, in order to check homological 1-connectedness over these fields,
4. "Lifting" tools to map fillings in "small" coset complexes to fillings in "large" coset complexes,

[8] Kaufman and Oppenheim [KO21] state that $p$ may be a prime power, but this is not completely clear because the Biss-Dasgupta result holds for the ring $\mathbb{Z}/q\mathbb{Z}$, but is different for the ring $\mathbb{F}_q$.

[9] We verified that it is false when $p = 3$, 5, or 7, by showing that the corresponding coset complexes are not simply connected. We conjecture that this is the case for all $p$ (see Conjecture 6.2 below). If our conjecture is wrong, our group-theoretic derivations could immediately be used to apply the absolute Dehn method.

5. Various group-theoretic derivations, which turn out to be quite elaborate, and were formalized in [WBCS25].

**Remark 1.21.** One interesting byproduct of the "lifting" technology we develop is that in the $A_3$ case, we can apply the *absolute* Dehn method to $\mathfrak{A}_q^3(m)$ for $q$ a (sufficiently large) power of $2, 5, 7, 11, 13$, using the results of [Kir78; Eve78] in place of [BD01]. $\Diamond$

### 1.4 Related work

After the initial version of our work appeared, Wang, Bhoja, Codel, and Singer [WBCS25] formalized the group-theoretic Theorem 5.5 in the Lean 4 theorem prover. In light of this, we have chosen to omit the original (very long) proof of Theorem 5.5 from this manuscript.

Concurrently with and independent of this paper, Kaufman, Oppenheim, and Weinberger [KOW25] proved that the *global* 1-cohomology vanishes in the $A_n$-type coset complexes $\mathfrak{A}_q^n(m)$ of Kaufman and Oppenheim [KO18] (even over arbitrary nonabelian coefficients). Combined with the proof from [KO21] of cosystolic expansion, this shows that $\mathfrak{A}_q^n(m)$ is a coboundary expander.

### 1.5 Outline

In §2, we define simplicial complexes, relevant notions of expansion, and (general) coset complexes, and cite some generic tools for proving expansion (i.e., local-to-global methods for cosystolic expansion [EK16; DD24a], and the cones method for coboundary expansion [Gro10; KO21]). In §3, we present our new tools for proving coboundary expansion of coset complexes (over arbitrary commutative coefficient rings), thereby establishing the relative Dehn method (Theorem 1.12). In §4, we then formally define Chevalley complexes and use tools from §3 to connect certain algebraic and computational conditions to expansion of these complexes. In §5, we use the aforementioned tools to prove our main application theorem (Theorem 1.19) regarding the $B_3$-type complexes of [OP22]. In §6, we discuss computational aspects of the proof. We emphasize that the tools for coboundary expansion in §3 work for all coset complexes; that section may be read independently of the latter sections and the methods may be of independent interest.

In Appendix A, we discuss the proof of simple connectivity in the $A_3$-type case due to [BD01; KO21] and reflect on why it does not appear to extend to the $B_3$-type case. In Appendix B, we reprove the cones method of [KO21] for general coefficient rings. In Appendix D, we formally define the $A_3$- and $B_3$-type groups which we discuss in the paper.

## 2 Preliminaries

We use the convention $[n] \coloneqq \{0, \ldots, n\}$, so that $|[n]| = n + 1$. (We warn the reader that this convention is nonstandard, since usually 0 is excluded, but it makes some notations significantly more convenient.) "Ring" means a commutative ring with identity. Without qualification, a "group" means a (possibly nonabelian) group, written multiplicatively.

### 2.1 Group theory

We denote group identity elements as $\mathbb{1}$, and define the commutator of two elements in a group $x, y \in G$ as:

$$[x, y] \coloneqq xyx^{-1}y^{-1}. \tag{2.1}$$

(We can similarly define the commutator of two words over $\mathcal{H}$ for $\mathcal{H}$ an indexed subgroup family in $G$.) Observe that $xy = yx$ (i.e., $x$ and $y$ commute) iff $[x, y] = \mathbb{1}$.

"Cosets" in this paper are always *left* cosets. For a group $G$ and a subgroup $H < G$, $G/H$ is the set of left cosets $\{gH : g \in G\}$. Here are some trivial but useful facts to keep in mind:

**Fact 2.2.** *Let $G$ be a group and $H < G$ a subgroup.*

- *For $g \in G$ and $h \in H$, we have $gH = ghH$.*
- *For every $g_1, g_2 \in G$, $g_1^{-1}g_2 \in H$ iff $g_1 \in g_2H$ iff $g_1H = g_2H$.*
- *For every $g_1, g_2 \in G$, $g_1^{-1}g_2 \notin H$ iff $g_1 \notin g_2H$ iff $g_1H \cap g_2H = \emptyset$.*

## 2.2 Simplicial complexes

Recall our definition of simplicial complexes (Definition 1.1) as given by vertices $V$, facets $\mathfrak{X}(d)$ (size-$(d+1)$ subsets of $V$), and a weighted distribution $\pi_d$ on facets. Here, we give several more notations and definitions regarding simplicial complexes which we will use in this paper.

**Definition 2.3.** For two vertices $u, v \in \mathfrak{X}(0)$, a *walk* of *length* $L \geq 0$ from $u$ to $v$ in $\mathfrak{X}$ is a sequence of vertices written $W = (u_0 = u) \to u_1 \to u_2 \to \cdots \to (u_L = v)$ such that for every $0 \leq \ell \leq L-1$, $\{u_\ell, u_{\ell+1}\} \in \mathfrak{X}(0) \cup \mathfrak{X}(1)$ (i.e., either $u_\ell = u_{\ell+1}$, or $u_\ell \neq u_{\ell+1}$ and $\{u_\ell, u_{\ell+1}\} \in \mathfrak{X}(1)$ is an edge).[10] We also use $|W|$ for the length of the walk. If for every $\ell$, $u_\ell \neq u_{\ell+1}$, we say $W$ is *proper*. Given a walk $W$ from $u$ to $v$ and a walk $W'$ from $v$ to $w$, $W \circ W'$ is the concatenated walk (from $u$ to $w$), and $|W \circ W'| = |W| + |W'|$. Given a walk $W$ from $u$ to $v$, $W^{-1}$ is the reversed walk (from $v$ to $u$). $\lozenge$

**Definition 2.4.** A walk from $u$ to $u$ is called a *loop* at $u$. ($u$ may be called the *base point* of the loop.) Reversing a loop or concatenating two loops also results in a loop. $\lozenge$

**Notation 2.5.** If we don't specify a weighting for $d$-dimensional complex $\mathfrak{X}$, then by default $\pi$ is assumed to be the uniform distribution on $\mathfrak{X}(d)$. $\lozenge$

**Definition 2.6.** For $\mathfrak{X}$ a $d$-dimensional simplicial complex and $\sigma \in \mathfrak{X}(j)$, the *link* of $\sigma$ is the $(d-j-1)$-dimensional simplicial complex with facets $\mathfrak{X}_\sigma(d-j-1) \coloneqq \{\tau \setminus \sigma : \tau \in \mathfrak{X}(d),\ \tau \supseteq \sigma\}$. If $\mathfrak{X}$ has weighting $\pi$, the link $X_\sigma$ becomes a weighted complex $(\mathfrak{X}_\sigma, \pi^\sigma)$ under the natural weighting induced by $\pi$, in which $\pi^\sigma(\gamma)$ is proportional to $\pi(\sigma \cup \gamma)$. $\lozenge$

**Definition 2.7.** A family $(\mathfrak{X}_m)_m$ of $d$-dimensional complexes is of *bounded degree* if there is some $D$ such that every vertex link $(\mathfrak{X}_m)_\sigma$ (meaning $|\sigma| = 1$) has cardinality at most $D$. $\lozenge$

**Definition 2.8.** For $\mathfrak{X}$ a $d$-dimensional complex, the set of *oriented $j$-faces* is the set

$$\vec{\mathfrak{X}}(j) \coloneqq \{(\sigma_0, \dots, \sigma_j) : \{\sigma_0, \dots, \sigma_j\} \in \mathfrak{X}(j)\}.$$

If $P : [j] \to [j]$ is a permutation and $\sigma = (\sigma_0, \dots, \sigma_j) \in \vec{\mathfrak{X}}(j)$, then we define $P(\sigma) \coloneqq (\sigma_{P(0)}, \dots, \sigma_{P(j)}) \in \vec{\mathfrak{X}}(j)$. If $\mathfrak{X}$ is weighted with facet distribution $\pi_d$, we naturally get a distribution $\vec{\pi}_j$ on oriented $j$-faces $\vec{\mathfrak{X}}(j)$ by sampling a $j$-face from $\pi_j$ and orienting it uniformly randomly. $\lozenge$

We identify $\mathfrak{X}(0)$ and $\vec{\mathfrak{X}}(0)$ (i.e., vertices need not be oriented).

**Definition 2.9.** Let $\mathfrak{X}$ be a $d$-dimensional (weighted) simplicial complex with uniform weighting on the facets. An *automorphism* of $\mathfrak{X}$ is a bijection $\varphi : \mathfrak{X}(0) \to \mathfrak{X}(0)$ such that the image of every facet is also a facet, i.e., for every $\sigma \in \mathfrak{X}(d)$, $\varphi(\sigma) \in \mathfrak{X}(d)$. We say $\mathfrak{X}$ is *strongly symmetric* if for all facets $\sigma, \sigma' \in \mathfrak{X}(d)$, there exists an automorphism $\varphi$ of $\mathfrak{X}$ such that $\varphi(\sigma) = \sigma'$. The automorphisms of $\mathfrak{X}$ form a group, denoted $\mathrm{Aut}(\mathfrak{X})$, under composition. $\lozenge$

**Fact 2.10.** *When $\mathfrak{X} = \mathfrak{CC}(G; \mathcal{H})$ is a coset complex and $g \in G$, the function $\varphi_g : \mathfrak{X}(0) \to \mathfrak{X}(0)$ defined by $\varphi_g(g' H_\lambda) \coloneqq (gg')H_\lambda$ is an automorphism of $\mathfrak{X}$, which we call* translation by $g$.

### 2.2.1 (Co)cycles and (co)boundaries

Let $\Gamma$ be a fixed ring.[11]

**Definition 2.11.** For $\mathfrak{X}$ a $d$-dimensional complex and $-1 \leq j \leq d$, a *$j$-chain* $f$ in $\mathfrak{X}$ is a function $f : \vec{\mathfrak{X}}(j) \to \Gamma$ satisfying the antisymmetry property $f(\sigma) = \mathrm{sign}(P) \cdot f(P(\sigma))$ for every permutation $P : [j] \to [j]$.[12] The set $C^j(\mathfrak{X}; \Gamma)$ of all $j$-chains forms an (abelian) group under (pointwise) addition. $\lozenge$

Note that a $j$-chain $f$ is completely determined by specifying $f$'s value on one orientation of every $j$-face of $\mathfrak{X}$. Hence, as a group $C^j(\mathfrak{X}; \Gamma)$ is isomorphic to the $|\mathfrak{X}(j)|$-fold direct sum of $\Gamma$ (though the isomorphism depends on choice of orientations).

Walks and loops are simple examples of 1-chains:

[10] In the context of simplicial complexes, this would usually be called a walk in the "1-skeleton" of the complex. However, we will not need any other notion of walks in this paper.

[11] We note that 1-cohomology can even be defined over nonabelian groups; see Appendix C.2.

[12] Arguably this should be called a *$j$-cochain*, but since our complexes are finite there is no need to distinguish.

**Definition 2.12.** Let $u, v \in \mathfrak{X}(0)$ and $\Gamma$ be a ring. Any (possibly improper) walk $W = (u_0 = u) \to \cdots \to (u_L = v)$ from $u$ to $v$ in $\mathfrak{X}$ gives rise to a 1-chain $[W]_\Gamma := \sum_{\ell : u_\ell \neq u_{\ell+1}} 1_{(u_\ell, u_{\ell+1})} \in C^1(\mathfrak{X}; \Gamma)$. $\Diamond$

Given an oriented $j$-face $\sigma = (\sigma_0, \ldots, \sigma_j)$ and $0 \le i \le j$, define the oriented $(j-1)$-face $\sigma_{\setminus i} := (\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_j)$.

**Definition 2.13.** For $\mathfrak{X}$ a $d$-dimensional complex and $0 \le j \le d$, the $(j-1)$-*coboundary operator* $\delta^{j-1} : C^{j-1}(\mathfrak{X}; \Gamma) \to C^j(\mathfrak{X}; \Gamma)$ is a homomorphism defined on $(j-1)$-chains $f$ by

$$(\delta^{j-1} f)(\sigma) := \sum_{i=0}^{j} (-1)^i f(\sigma_{\setminus i})$$

for every $\sigma \in \vec{\mathfrak{X}}(j)$. Dually, the $j$-boundary operator $\partial_j : C^j(\mathfrak{X}; \Gamma) \to C^{j-1}(\mathfrak{X}; \Gamma)$ is the homomorphism defined by: For $\sigma \in \vec{\mathfrak{X}}(j)$, $g \in \Gamma$, and $1^g_\sigma$ the 1-chain sending $\sigma$ to $g$ and all other faces to 0,

$$\partial_j 1^g_\sigma = \sum_{i=0}^{j} (-1)^i 1^g_{\sigma_{\setminus i}}.$$

$\Diamond$

These satisfy that $\delta^{j-1}\delta^j = \partial_j \partial_{j-1} = 0$. Given a fixed choice of orientations for $(j-1)$- and $j$-faces, $\delta^{j-1}$ and $\partial_j$ can be represented as $\{0, +1, -1\}$-valued matrices (without regard to the choice of $\Gamma$).

**Remark 2.14.** If $\Gamma = \mathbb{Z}_2$, since $1 = -1$, the antisymmetry property of chains becomes the symmetry property $f(\sigma) = f(P(\sigma))$. Hence, chains over $\mathbb{Z}_2$ can more simply be viewed as functions $\mathfrak{X}(j) \to \mathbb{Z}_2$, or equivalently, as (indicators of) sets of $j$-faces. Moreover, the coboundary operator can be written more simply as $\delta^{j-1} f(\sigma) = \sum_{\tau \supset \sigma} f(\tau)$. This recovers the 1-dimensional definitions we gave in §1.1. $\Diamond$

**Definition 2.15.** A *$j$-cycle* is a $j$-chain $f \in C^j(\mathfrak{X}; \Gamma)$ such that $\partial_j f = 0$. In particular, $f \in C^1(\mathfrak{X}; \Gamma)$ is a 1-cycle iff for every vertex $u \in \mathfrak{X}(0)$,

$$\sum_{v : (u,v) \in \vec{\mathfrak{X}}(1)} f(u, v) = 0.$$ [13]

Over any $\Gamma$, a walk $W$ from $u$ to $v$ has 1-boundary $\partial_1 [W]_\Gamma = 1_v - 1_u$, and in particular, if $W$ is a loop at $u$ then $[W]_\Gamma$ is a 1-cycle. The $j$-cycles form a subgroup $Z_j(\mathfrak{X}; \Gamma) := \ker \partial_j \subseteq C^j(\mathfrak{X}; \Gamma)$. $\Diamond$

Similarly:

**Definition 2.16.** The group of *$j$-boundaries* is $B_j(\mathfrak{X}; \Gamma) := \operatorname{im} \partial_{j-1} \subseteq C^j(\mathfrak{X}; \Gamma)$. If $f$ is a 1-boundary and $T$ is a 2-chain such that $\partial_2 T = f$, then we say $T$ is a $\Gamma$-*filling* of $f$. Since $\delta^j \delta^{j-1} = 0$, the $j$-boundaries are always "trivially" $j$-cycles, i.e., $B_j(\mathfrak{X}; \Gamma) \subseteq Z_j(\mathfrak{X}; \Gamma)$. $\Diamond$

Dually:

**Definition 2.17.** A *$j$-cocycle* is a $j$-chain $f \in C^j(\mathfrak{X}; \Gamma)$ such that $\delta^j f = 0$. In particular, a 1-*cochain* $f \in C^1(\mathfrak{X}; \Gamma)$ satisfies that for every oriented triangle $(u, v, w) \in \vec{\mathfrak{X}}(2)$,

$$f(u, v) + f(v, w) + f(w, u) = 0.$$

The $j$-cocycles form a subgroup $Z^j(\mathfrak{X}; \Gamma) := \ker \delta^j \subseteq C^j(\mathfrak{X}; \Gamma)$. Similarly, the group of $j$-coboundaries is $B^j(\mathfrak{X}; \Gamma) := \operatorname{im} \delta^{j-1} \subseteq Z^j(\mathfrak{X}; \Gamma) \subseteq C^j(\mathfrak{X}; \Gamma)$. $\Diamond$

**Definition 2.18.** The *$j$-th cohomology group* (over $\Gamma$) is the quotient group $H^j(\mathfrak{X}; \Gamma) := Z^j(\mathfrak{X}; \Gamma)/B^j(\mathfrak{X}; \Gamma)$. Dually, the *$j$-th homology group* is the quotient group $H_j(\mathfrak{X}; \Gamma) := Z_j(\mathfrak{X}; \Gamma)/B_j(\mathfrak{X}; \Gamma)$. The complex $\mathfrak{X}$ is *homologically $j$-connected over* $\Gamma$ (or "has vanishing $j$-homology over $\Gamma$") if any if the equivalent conditions $B^j(\mathfrak{X}; \Gamma) = Z^j(\mathfrak{X}; \Gamma)$, $H^j(\mathfrak{X}; \Gamma) = 0$, $B_j(\mathfrak{X}; \Gamma) = Z_j(\mathfrak{X}; \Gamma)$, or $H_j(\mathfrak{X}; \Gamma) = 0$ hold. $\Diamond$

**Definition 2.19.** When $\Gamma$ is a *field*, $C^j(\mathfrak{X}; \Gamma), Z^j(\mathfrak{X}; \Gamma), B^j(\mathfrak{X}; \Gamma), H^j(\mathfrak{X}; \Gamma)$ and their dual (homology) versions are all vector spaces over $\Gamma$. Similarly, $\delta^j$ and $\partial_j$ are linear maps. In this case, we define the *$j$-th Betti number* of $\mathfrak{X}$ over $\Gamma$ as

$$\beta_j(\mathfrak{X}; \Gamma) := \dim(H^j(\mathfrak{X}; \Gamma); \Gamma) = \dim(H_j(\mathfrak{X}; \Gamma); \Gamma).$$

In particular, $\mathfrak{X}$ is homologically $j$-connected over $\Gamma$ iff $\beta_j(\mathfrak{X}; \Gamma) = 0$. $\Diamond$

[13]In spirit, such $f$ may be thought of as a "net-zero flow", though $f$'s values are in $\Gamma$, not necessarily real numbers.

**Fact 2.20.** *The following conditions are equivalent:*

1. *$\mathfrak{X}$ is 0-connected,*
2. *$\mathfrak{X}$ is 0-connected over every $\Gamma$, and*
3. *$\mathfrak{X}$ is 0-connected over some $\Gamma$ (with $|\Gamma| \geq 2$).*

*When $\mathfrak{X}$ is 0-connected, we define $\mu(\mathfrak{X}) \coloneqq |\mathfrak{X}(1)| - |\mathfrak{X}(0)| + 1$ as the* 1-skeletal cyclomatic number *of $\mathfrak{X}$.*

This gives us the following characterization of homological 1-connectivity:

**Fact 2.21.** *If $\mathfrak{X}$ is 0-connected, then $\beta_1(\mathfrak{X};\Gamma) = \mu(\mathfrak{X}) - \operatorname{rank}(\delta^1;\Gamma)$. Hence, $\mathfrak{X}$ is homologically 1-connected over $\Gamma$ iff $\operatorname{rank}(\delta^1;\Gamma) = \mu(\mathfrak{X})$.*

*Proof.* By definition, $\beta_1(\mathfrak{X};\Gamma) = |\mathfrak{X}(1)| - (\operatorname{rank}(\delta^0;\Gamma) + \operatorname{rank}(\delta^1;\Gamma))$. By 0-connectivity, $\operatorname{rank}(\delta^0;\Gamma) = |\mathfrak{X}(0)| - 1$. This gives the required bound. ☐

**Remark 2.22.** We stress that (at least when $\Gamma$ is $\mathbb{F}_p$ or $\mathbb{Z}$), homological $j$-connectivity of an explicit simplicial complex is checkable in polynomial time, via computing the ranks of the boundary operators $\delta^{j-1}$ and $\delta^j$. In contrast, checking *simple* connectivity of an explicit simplicial complex — the stronger notion of connectivity established by the absolute Dehn method of [KO21] — is equivalent to checking triviality of a finitely presented group, which is undecidable [Adi57; Rab58]. (In particular, given an explicit finitely presented group, one can compute an explicit simplicial complex which is simply connected iff the group is trivial, e.g., the second barycentric subdivision of the presentation complex.) ◊

## 2.3 Expansion definitions

In this section on expansion properties, $(\mathfrak{X}, \pi)$ will always denote a weighted $d$-dimensional simplicial complex.

### 2.3.1 Cocycle, coboundary, and cosystolic expansion

In this section, fix a ring $\Gamma$.

**Definition 2.23.** For $(\mathfrak{X}, \pi)$ a weighted $d$-dimensional complex, the *distance* between two $j$-chains $f$ and $f'$ is defined to be

$$\operatorname{dist}(f, f') \coloneqq \Pr_{\boldsymbol{\sigma} \sim \pi_j}[f(\boldsymbol{\sigma}) \neq f'(\boldsymbol{\sigma})].$$

The *weight* of a $j$-chain is $\operatorname{wt}(f) \coloneqq \operatorname{dist}(f, 0)$. ◊

Note that we need not specify an orientation for $\boldsymbol{\sigma}$ since $f(\boldsymbol{\sigma}) = f'(\boldsymbol{\sigma})$ iff $f(P(\boldsymbol{\sigma})) = f'(P(\boldsymbol{\sigma}))$ by antisymmetry.

**Definition 2.24.** Suppose that whenever $f$ is a $j$-chain over $\Gamma$ with $\operatorname{dist}(f, Z^j(\mathfrak{X};\Gamma)) > v$, it holds that $\operatorname{dist}(\delta^f, 0) > \epsilon \cdot v$. Then we say that $\mathfrak{X}$ has *$j$-cocycle expansion (at least) $\epsilon$ over* $\Gamma$, and we write $h^j(\mathfrak{X};\Gamma)$ for the least possible such $\epsilon$. If the "$j$" is omitted, we mean that the condition holds for all $0 \leq j < d$. ◊

**Definition 2.25.** We define the *$j$-cosystole* to be

$$s^j(\mathfrak{X};\Gamma) \coloneqq \min\{\operatorname{dist}(f, 0) : b \in Z^j(\mathfrak{X};\Gamma) \setminus B^j(\mathfrak{X};\Gamma)\}, \tag{2.26}$$

with the convention $s^j(\mathfrak{X};\Gamma) = 1$ if the $j$-th cohomology vanishes over $\Gamma$. ◊

**Definition 2.27.** Suppose the $j$-cocycle expansion over $\Gamma$ satisfies $h^j(\mathfrak{X};\Gamma) \geq \epsilon$. If, moreover, $s^j(\mathfrak{X};\Gamma) \geq \mu$, we say that $\mathfrak{X}$ has *$j$-cosystolic expansion (at least) $(\epsilon, \mu)$ over* $\Gamma$. If the $j$-th cohomology in fact vanishes, $\mathfrak{X}$ is said to have *$j$-coboundary expansion (at least) $\epsilon$*. If the "$j$" is omitted, we mean that the condition holds for all $0 \leq j < d$. ◊

### 2.3.2 Spectral expansion

**Definition 2.28.** Let $\sigma \in \mathfrak{X}(j)$, $j \leq d-2$, and let $G_\sigma = (V, E, \pi)$ be the 1-skeleton of the link $\mathfrak{X}_\sigma$. Then if $A_\sigma$ denotes the ($\pi_1$-weighted) adjacency matrix for $G_\sigma$, and $D_\sigma$ denotes the diagonal degree matrix (with $v$th diagonal entry $2\pi_0(\{v\})$), we write $W_\sigma = D_\sigma^{-1} A_\sigma$ for the *standard random walk matrix of* $\mathfrak{X}_\sigma$ (which has invariant distribution $\pi_0$). We will also write $\lambda_2(\mathfrak{X}_\sigma)$ for the second-largest eigenvalue of $W_\sigma$. $\Diamond$

**Definition 2.29.** For $j \leq d-1$, we say that $\mathfrak{X}$ has *$j$-spectral*[14] *expansion parameter at most* $\gamma$ if $\lambda_2(\mathfrak{X}_\sigma) \leq \gamma$ for all $|\sigma| = j$. If the "$j$" is omitted, we mean that the condition holds for all $j \leq d-2$. $\Diamond$

Ballmann and Świątkowski [BŚ97] showed the following:

**Theorem 2.30.** *Suppose $\mathfrak{X}$ has 1-spectral expansion parameter $\gamma$, and also that (the 1-skeleton of) $\mathfrak{X}$ is connected. Then $\lambda_2(\mathfrak{X}_\emptyset) \leq \frac{\gamma}{1-\gamma}$; i.e., the 0-spectral expansion parameter of $\mathfrak{X}$ is at most $\frac{\gamma}{1-\gamma}$.*

Using this inductively, one can establish [Opp18] the *Trickling Down* theorem:

**Theorem 2.31.** *Assume (the 1-skeleton of) $\mathfrak{X}_\sigma$ is connected for all $j$-faces $\sigma$, $j < d-1$. If $\mathfrak{X}$ has $(d-1)$-spectral expansion parameter $\gamma$, then it has $j$-spectral expansion parameter at most $\frac{\gamma}{1-(d-1-j)\gamma}$.*

The utility of this theorem is that (under the mild constraint of connectivity) one can show *global* spectral expansion just by verifying spectral expansion in the *local* link graphs of the $(d-2)$-dimensional faces.

## 2.4 Cosystolic expansion from local conditions

A sequence of works [KKL16; EK16; DD24a] gave results showing that cosystolic expansion — and hence also topological expansion — follows from "local" considerations. The following version is taken from [DD24a]:

**Theorem 2.32** ([DD24a, Thm. 1.2])**.** *For any $0 < \beta < \frac{1}{2}$ and $d \in \mathbb{N}$, there are $\gamma, \epsilon, \mu > 0$ such that the following holds. Let $\Gamma$ be a group and $\mathfrak{X}$ a $d$-dimensional simplicial complex. Suppose that for all $j$-faces $\sigma$, $0 \leq j < d-1$, the complex $\mathfrak{X}_\sigma$ has coboundary expansion at least $\beta$ over $\Gamma$. Suppose also that $\mathfrak{X}$ has spectral expansion parameter at most $\gamma$. Then $\mathfrak{X}$ has $j$-cosystolic expansion at least $(\epsilon, \mu)$ for all $j < d-1$ (though not necessarily for $j = d-1$).*

Suppose we wish to apply this theorem in the case $d = 3$. In terms of coboundary expansion, we would need to verify:

1. Each vertex-link has 1-coboundary expansion at least $\beta$.
2. Each vertex-link has 0-coboundary expansion at least $\beta$.
3. Each edge-link has 0-coboundary expansion at least $\beta$.

But the second and third conditions here are essentially superfluous, since $\mathfrak{X}$ is already assumed to be an excellent spectral expander. More precisely, by the "easy direction" of Cheeger's inequality, the 0-coboundary expansion of the vertex- and edge-links is at least $\frac{1}{2} - \frac{1}{2}\gamma$, which exceeds $\beta$ provided $\gamma$ is small enough. Thus only the first condition above is essential. Moreover, by the Trickling Down theorem, we only need to verify spectral expansion for the edge-links. Putting this together (exactly as was done in [KO21]), we conclude the following:

**Theorem 2.33** (Local-to-global + Trickling Down)**.** *For any $0 < \beta < \frac{1}{2}$, there exists $\gamma > 0$ such that the following holds: Suppose that $\mathfrak{X}$ is a connected 3-dimensional complex, where all vertex-links are connected, and all edge-links have 0-spectral expansion parameter (i.e., second-largest eigenvalue) at most $\gamma$. Moreover, let $\Gamma$ be any group, and assume that each vertex-link has coboundary expansion at least $\beta$ over $\Gamma$. Then $\mathfrak{X}$ has 1-cosystolic expansion at least $(\Omega(\beta), \Omega(\beta))$ over $\Gamma$.*

[14]Other authors would call this $(j-1)$-local or $(j-1)$-spectral expansion.

## 2.5 Cones and fillings

The main effort is in developing a way to show good 1-coboundary expansion for certain 2-dimensional complexes. One way to show this is to use the "random cones" method of Gromov [Gro10]. In turn, Kaufman and Oppenheim [KO21] (following on ideas in [LMM16; KM19]) showed that in "strongly symmetric" complexes, it suffices to upper-bound the "cone radius". We state a version of their result which suffices for our purposes:

**Definition 2.34.** Let $\mathfrak{X}$ be a $d$-dimensional simplicial complex ($d \geq 2$), $\Gamma$ a ring, and $L$ a loop in $\mathfrak{X}$. We define the *("homology") area* of $L$ (over $\Gamma$) as:

$$\triangle_{\mathrm{H}}^{\Gamma}(L) \coloneqq \min_{\substack{T \in C^2(\mathfrak{CC}(G;\mathcal{H});\Gamma),\\ \partial_2 T = [L]_\Gamma}} |\mathrm{supp}(T)|.$$

(We call such $T$ a $\Gamma$-*filling* of $L$.) We write $\triangle_{\mathrm{H}}^{\Gamma}(L) = \infty$ if there is no such $T$. $\Diamond$

We outline some important properties of the function $\triangle_{\mathrm{H}}^{\Gamma}$ below in §3.1.

**Definition 2.35.** Let $\mathfrak{X}$ be a $d$-dimensional simplicial complex ($d \geq 2$) and $\Gamma$ a ring. We say that $\mathfrak{X}$ is $(R_0, R_1)$-*homologically taut over* $\Gamma$ if for every proper loop $L$ of length at most $R_0$, $\triangle_{\mathrm{H}}^{\Gamma}(L) \leq R_1$. $\Diamond$

(This definition of tautness generalizes the hypothesis of the cones method in [KO21].)

Kaufman and Oppenheim's theorem makes this quantitative for strongly symmetric complexes:

**Theorem 2.36** (Cones method, generalizing [KO21, Thm. 3.8])**.** *Let $\Gamma$ be a ring and let $\mathfrak{X}$ be a 2-dimensional simplicial complex. Assume that $\mathfrak{X}$ is strongly symmetric (and therefore $\pi_2$ is the uniform distribution on $\mathfrak{X}(2)$). If $R_0$ is the diameter of $\mathfrak{X}$ and $\mathfrak{X}$ is $(2R_0 + 1, R_1)$-homologically taut over $\Gamma$, then $\mathfrak{X}$ has 1-coboundary expansion at least $1/R_1$ over $\Gamma$.*

Since this theorem is not stated for general rings $\Gamma$ in [KO21], we reprove it in our Appendix B. (We also reprove a "homotopy" version of it due to [DD24b], implying expansion over all *nonabelian* groups, in Appendix C.3.)

## 2.6 Coset complexes

Let $G$ be a group and $\mathcal{H} = (H_\lambda < G)_{\lambda \in \Lambda}$ a $\Lambda$-indexed subgroup family. We collect some elementary properties of a coset complexes here (see, e.g., [Gar79] for proofs):

**Proposition 2.37.** *The $|\Lambda| - 1$-dimensional coset complex $\mathfrak{CC}(G;\mathcal{H})$ satisfies the following:*

1. *For every $0 \leq j \leq |\Lambda| - 1$, the group $G$ acts on $\mathfrak{CC}(G;\mathcal{H})$ as automorphisms in a natural way: The $j$-face $\{xH_{\lambda_0}, \ldots, xH_{\lambda_j}\}$ is mapped to the $j$-face $\{gxH_{\lambda_0}, \ldots, gxH_{\lambda_j}\}$. This action is transitive on the $j$-faces.*
2. *Let $\sigma$ be a $j$-face in $\mathfrak{CC}(G;\mathcal{H})$; say $\sigma = \{xH_{\lambda_0}, \ldots, xH_{\lambda_j}\}$ where $\lambda_0, \ldots, \lambda_j \in \Lambda$ are all distinct. Let $\Lambda' \coloneqq \{\lambda_0, \ldots, \lambda_j\}$. Then writing $H_{\Lambda'} = \bigcap_{\lambda \in \Lambda'} H_\lambda$ (meaning $G$ itself if $\Lambda' = \emptyset$), the link of $\sigma$ is isomorphic to the coset complex $\mathfrak{CC}\big(H_{\Lambda'}; (H_{\Lambda'} \cap H_\lambda)_{\lambda \in \Lambda \setminus \Lambda'}\big)$.*

# 3 The relative Dehn method

In this section, we develop technical notions in order to prove our relative Dehn method, Theorem 1.12.

## 3.1 Area function

We start by defining an abstract notion of "area functions" that captures the properties of the functions $\triangle_{\mathrm{H}}^{\Gamma}$ that we will use.

**Definition 3.1.** Let $\mathfrak{X}$ be a simplicial complex. A *backtracking loop* at a vertex $u \in \mathfrak{X}(0)$ is a loop of the form $W \circ W^{-1}$ or $W \circ (v \to v) \circ W^{-1}$ where $W$ is a walk from $u$ to $v$ (for some vertex $v \in \mathfrak{X}(0)$). $\Diamond$

**Definition 3.2** (Area function)**.** Let $\mathfrak{X}$ be a simplicial complex. A function $\triangle : \{\text{loops in } \mathfrak{X}\} \to \mathbb{N} \cup \{\infty\}$ is an *area function* if it satisfies the following axioms:

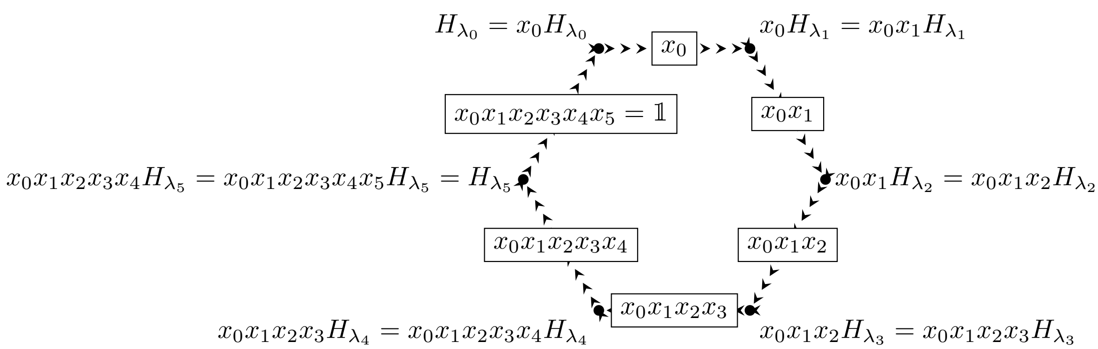



Figure 1: Suppose $\widehat{w} = \binom{x_0}{\lambda_0}\binom{x_1}{\lambda_1}\binom{x_2}{\lambda_2}\binom{x_3}{\lambda_3}\binom{x_4}{\lambda_4}\binom{x_5}{\lambda_5} \in F_{\mathcal{H}}$ is a length-6 colored word with $\varphi(\widehat{w}) = \mathbb{1}$ (equivalently, $x_0x_1x_2x_3x_4x_5 = \mathbb{1} \in G$). This figure depicts the corresponding cycle $\mathcal{L}(\widehat{w})$. The vertices are cosets; note the alternative names given to each vertex (corresponding to different representatives of each vertex.). The edges are labeled with elements "witnessing" the intersection of the cosets corresponding to the two incident vertices.

1. *Cyclic symmetry:* If $W$ is a walk from $u$ to $v$ and $W'$ from $v$ to $u$, then $\triangle(W \circ W') = \triangle(W' \circ W)$.
2. *Reversal symmetry:* For every loop $L$, $\triangle(L) = \triangle(L^{-1})$.
3. *Translation symmetry:* For every loop $L$ and automorphism $\varphi \in \mathrm{Aut}(\mathfrak{X})$, $\triangle(\varphi L) = \triangle(L)$.
4. *Backtracking loops have no area:* For every backtracking loop $B$, $\triangle(B) = 0$.
5. *Triangles have unit area:* If $\{u, v, w\} \in \mathfrak{X}(2)$, then $\triangle((u \to v \to w \to u)) = 1$.
6. *Subadditivity:* Suppose $W_1$ is a walk from $u$ to $v$, $W_2$ a walk from $v$ to $u$, and $L$ a loop at $v$. Then $\triangle(W_1 \circ L \circ W_2) \leq \triangle(L) + \triangle(W_1 \circ W_2)$. ◊

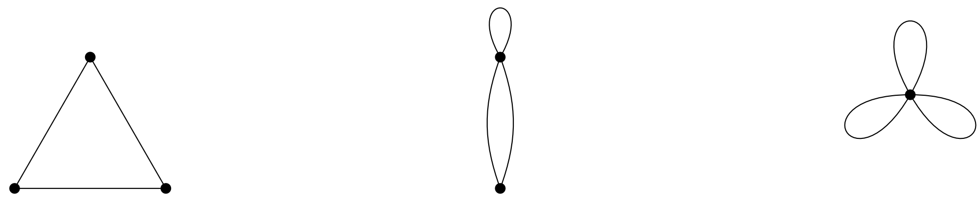
**Proposition 3.3.** *For every ring $\Gamma$ and simplicial complex $\mathfrak{X}$, the area function $\triangle_{\mathrm{H}}^{\Gamma}$ defined in Definition 2.34 satisfies the axioms in Definition 3.2.*

*Proof.* Recall that for a loop $L$, $\triangle_{\mathrm{H}}^{\Gamma}(L) = \min\{|\mathrm{supp}(T)| : \partial_2 T = [L]_\Gamma\}$. Cyclic symmetry follows from the fact that $[W \circ W']_\Gamma = [W]_\Gamma + [W']_\Gamma$. Reversal symmetry follows from the fact that $[L^{-1}]_\Gamma = -[L]_\Gamma$ (and $\partial_2(-T) = -\partial_2 T$). Translation symmetry follows from the fact that $[\varphi L]_\Gamma = \varphi[L]_\Gamma$ and $\partial_2(\varphi T) = \varphi(\partial_2 T)$. The backtracking walk property follows from the fact that $[W \circ W^{-1}]_\Gamma = [W]_\Gamma + [W^{-1}]_\Gamma = [W]_\Gamma - [W]_\Gamma = 0$, and similarly $[W \circ (v \to v) \circ W^{-1}]_\Gamma = 0$ since $[(v \to v)]_\Gamma = 0$ by definition. The unit area property follows from the fact that $\partial_2 1_{(u,v,w)} = 1_{(u,v)} + 1_{(v,w)} + 1_{(w,u)} = [(u, v, w)]_\Gamma$. Finally, subadditivity follows from the fact that if $\partial_2 T = [L]_\Gamma$ and $\partial_2 T' = [L']_\Gamma$, then $\partial_2(T + T') = [L \circ L']_\Gamma$ and $|\mathrm{supp}(T + T')| \leq |\mathrm{supp}(T)| + |\mathrm{supp}(T')|$. ☐

## 3.2 From colored words to loops

In this subsection, fix a finite group $G$, an indexed subgroup family $\mathcal{H} = (H_\lambda < G)_{\lambda \in \Lambda}$, and consider the corresponding coset complex $\mathfrak{CC}(G; \mathcal{H})$. We also fix an area function $\triangle$. Recall that for any $g \in G$ and distinct pair of subgroup indices $\lambda_1 \neq \lambda_2 \in \Lambda$, the vertices $gH_{\lambda_1}$ and $gH_{\lambda_2}$ form an *edge* in $\mathfrak{CC}(G; \mathcal{H})$. Recall also the definition of a loop in a simplicial complex (Definition 2.3) and how a loop can be interpreted as a 1-chain (Definition 2.12).

**Definition 3.4.** Think of each subgroup index $\lambda \in \Lambda$ as a "color", and for any $x \in H_\lambda$, we introduce the symbol $\binom{\lambda}{x}$, called a "colored element". A *colored word* is a sequence of the form $\binom{\lambda_0}{x_0} \cdots \binom{\lambda_{\ell-1}}{x_{\ell-1}}$, where each $\binom{\lambda_i}{x_i}$ is a colored element. We regard this colored word as a *coloring* of the underlying word $\langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ over $\mathcal{H}$. By definition, every word $\langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ over $\mathcal{H}$ admits at least one coloring $\binom{\lambda_0}{x_0} \cdots \binom{\lambda_{\ell-1}}{x_{\ell-1}}$. ◊



(a) "Non-degenerate triangle" (three distinct vertices).
(b) "Partially degenerate triangle" (two distinct vertices).
(c) "Fully degenerate triangle" (one distinct vertex).

Figure 2: The three possible types of walks of length-3, allowing vertices to coincide.

**Definition 3.5.** Let $w = \langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ be a word over $\mathcal{H}$ of length $\ell$ and $\widehat{w} = \binom{\lambda_0}{x_0}\binom{\lambda_1}{x_1}\cdots\binom{\lambda_{\ell-1}}{x_{\ell-1}}$ a coloring of $\widehat{w}$.. (We typically use a $\widehat{\text{hat}}$ to denote a coloring of a word.) We associate to $\widehat{w}$ the following walk $\mathcal{L}(\widehat{w})$ of length $\ell$ in $\mathfrak{CC}(G; \mathcal{H})$:

$$\mathbb{1}H_{\lambda_0} \to x_0 H_{\lambda_1} \to x_0 x_1 H_{\lambda_2} \to \cdots \to x_0 x_1 \cdots x_{\ell-2} H_{\lambda_{\ell-1}} \to x_0 x_1 \cdots x_{\ell-1} H_{\lambda_0} = \mathsf{eval}(w) H_{\lambda_0}, \tag{3.6}$$

which is a loop iff $w$ is a relator (i.e., $\mathsf{eval}(w) = \mathbb{1}$). The walk $\mathcal{L}(\widehat{w})$ is proper iff $\lambda_i \neq \lambda_{i+1}$ for every $0 \le i \le \ell - 1$ (where $\lambda_\ell \coloneqq \lambda_0$). ◊

**Remark 3.7.** Let us remark that in Equation (3.6), wherever we wrote $x_0 \cdots x_{i-1} H_{\lambda_i}$, we could have equally well written $x_0 \cdots x_{i-1} x_i H_{\lambda_i}$; this is because $x_i \in H_{\lambda_i}$, so $x_i H_{\lambda_i} = H_{\lambda_i}$ (cf. Fact 2.2). So for example, the presence of the edge between $x_0 \cdots x_{i-1} H_{\lambda_i} = x_0 \cdots x_{i-1} x_i H_{\lambda_i}$ and $x_0 \cdots x_i H_{\lambda_{i+1}}$ in the skeleton of $\mathfrak{CC}(G; \mathcal{H})$ is "witnessed" by the element $x_0 \cdots x_i$. See Figure 1 for a visual depiction. ◊

**Remark 3.8.** Consideration of the above definition lets one easily conclude Proposition 1.6 from Proposition 2.37, that $\mathfrak{CC}(G; \mathcal{H})$ is connected iff $G$ is generated by the subgroups $\mathcal{H}$ iff $\mathsf{eval}$ is onto. ◊

**Definition 3.9.** When $w$ is a relator and $\widehat{w}$ is a coloring thereof, we write $\triangle(\widehat{w}) \coloneqq \triangle(\mathcal{L}(\widehat{w}))$. ◊

We now give some simple facts about areas.

**Fact 3.10.** *For any $\lambda \in \Lambda$, the length-1 word $\widehat{w} = \binom{\mathbb{1}}{\lambda}$ has $\triangle(\widehat{w}) = 0$.*

*Proof.* $\mathcal{L}(\widehat{w})$ is the self-loop at the vertex $\mathbb{1}H_\lambda$; use the backtracking axiom. □

**Fact 3.11.** *For any $\lambda_1, \lambda_2 \in \Lambda$ and $s \in H_{\lambda_1} \cap H_{\lambda_2}$, the length-2 "recoloring" word $\widehat{w} = \binom{s}{\lambda_1}\binom{s^{-1}}{\lambda_2}$ has $\triangle(\widehat{w}) = 0$.*

*Proof.* $\mathcal{L}(w)$ is a length-2 closed walk; use the backtracking axiom. □

**Fact 3.12.** *Let $w = \langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ be a relator over $\mathcal{H}$ admitting a monochromatic coloring $\widehat{w}$, i.e., there exists $\lambda \in \Lambda$ such that $x_0, \dots, x_{\ell-1} \in H_\lambda$, and $\widehat{w} \coloneqq \binom{\lambda}{x_0}\binom{\lambda}{x_1}\cdots\binom{\lambda}{x_{\ell-1}}$. Then $\triangle(\widehat{w}) = 0$.*

*Proof.* $\mathcal{L}(\widehat{w})$ consists of $\ell$ self-loops at $\mathbb{1}H$; again, use the backtracking axiom. □

**Fact 3.13.** *Let $w = \langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ be a relator over $\mathcal{H}$. Then for every coloring $\widehat{w} = \binom{\lambda_0}{x_0}\binom{\lambda_1}{x_1}\cdots\binom{\lambda_{\ell-1}}{x_{\ell-1}}$ of $w$, $\triangle(\widehat{w}^{-1}) = \triangle(\widehat{w})$.*

*Proof.* $x_0 x_1 \cdots x_{j-1} \mathcal{L}(\widehat{w}^j) = \mathcal{L}(\widehat{w})$, and hence $\mathcal{L}(\widehat{w}^{-1})$ is the reverse of $\mathcal{L}(\widehat{w})$; use the reversal symmetry property. □

**Fact 3.14.** *Let $w = \langle x_0 \rangle \cdots \langle x_{\ell-1} \rangle$ be a relator over $\mathcal{H}$. If $\widehat{w}^j = \binom{\lambda_j}{x_j}\binom{\lambda_{j+1}}{x_{j+1}}\cdots\binom{\lambda_{\ell-1}}{x_{\ell-1}}\binom{\lambda_0}{x_0}\cdots\binom{\lambda_{j-1}}{x_{j-1}}$ is some cyclic shift of $\widehat{w}$, then $\triangle(\widehat{w}^j) = \triangle(\widehat{w})$.*

*Proof.* $\mathcal{L}(\widehat{w}^j)$ is a cyclic shift of $\mathcal{L}(\widehat{w})$; use the cyclic symmetry property. □

See Figure 2 for a visual depiction of the following fact:

**Fact 3.15.** *Let $g \in G$ and $L = (gH_{\lambda_0}, gH_{\lambda_1}, gH_{\lambda_2})$. Then $\triangle(\widehat{w}) \le 1$.*

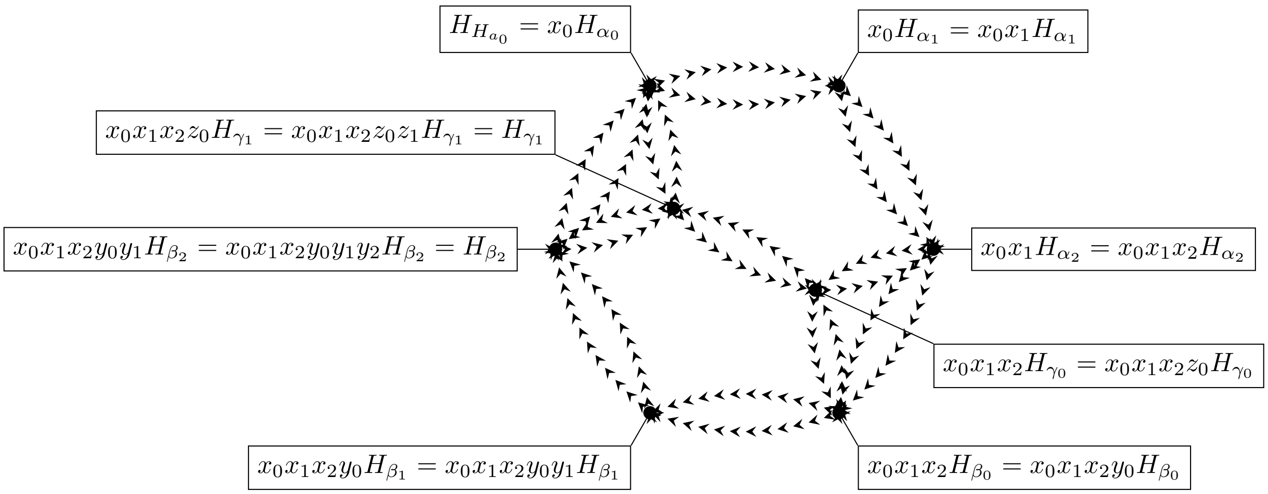



Figure 3: A "proof by picture" of Proposition 3.16. Let $\widehat{x} = \binom{\alpha_0}{x_0}\binom{\alpha_1}{x_1}\binom{\alpha_2}{x_2}$, $\widehat{y} = \binom{\beta_0}{y_0}\binom{\beta_1}{y_1}\binom{\beta_2}{y_2}$, and $\widehat{z} = \binom{\gamma_0}{z_0}\binom{\gamma_1}{z_1} \in F_{\mathcal{H}}$. Suppose also that $\mathsf{eval}(\widehat{x}\widehat{y}) = 1$ and $\mathsf{eval}(\widehat{y}) = \mathsf{eval}(\widehat{z})$. The goal is to reduce the problem of filling $\mathcal{L}(\widehat{x}\widehat{z})$ to filling $\mathcal{L}(\widehat{x}\widehat{y})$. The outer (rounded) hexagon is $\mathcal{L}(\widehat{x}\widehat{y})$ (compare Figure 1). We introduce two new vertices in the middle of the hexagon and draw two triangles and two pentagons involving these vertices. These four shapes add together to form the solid hexagon. The lower pentagon is the translation (by $\mathsf{eval}(\widehat{x})$) of $\mathcal{L}(\widehat{y}\widehat{z}^{-1})$ and the upper pentagon is $\mathcal{L}(\widehat{x}\widehat{z})$.

*Proof.* If $\lambda_0$, $\lambda_1$, and $\lambda_2$ are not all distinct, $L$ is either a triple self-loop or a backtracking edge plus a self loop; in either case, $L$ has area zero. Otherwise, $\{gH_{\lambda_0}, gH_{\lambda_1}, gH_{\lambda_2}\}$ is a *bona fide* triangle in $\mathfrak{X}$, i.e., an element of $\mathfrak{X}(2)$; consequently, $L$ has area one. (See Figure 2 for a graphical depiction.) $\square$

The following proposition is important; informally, it says that if "$xy \equiv \mathbb{1}$" has a small filling, and "$y \equiv z$" has a small filling, then we can deduce that "$xz \equiv \mathbb{1}$" has a small filling.

**Proposition 3.16.** *Let $x$, $y$, and $z$ be words over $\mathcal{H}$, and suppose that $xy$ and $yz^{-1}$ are both relators (so that $xz$ is also a relator). Then for all colorings $\widehat{x}$, $\widehat{y}$, and $\widehat{z}$ of $x$, $y$, and $z$, respectively, $\triangle(\widehat{x}\widehat{z}) \leq \triangle(\widehat{x}\widehat{y}) + \triangle(\widehat{y}\widehat{z}^{-1}) + 2$.*

See Figure 3 for a graphical depiction of the proof.

*Proof.* We will assume $x$, $y$, and $z$ all have length at least 1, as the case when one or more has length 0 is easier. Let $x = \langle x_0\rangle \cdots \langle x_{j-1}\rangle$, $y = \langle y_0\rangle \cdots \langle y_{k-1}\rangle$, and $z = \langle z_0\rangle \cdots \langle z_{\ell-1}\rangle$, and $\widehat{x} = \binom{\alpha_0}{x_0} \cdots \binom{\alpha_{j-1}}{x_{j-1}}$, $\widehat{y} = \binom{\beta_0}{y_0} \cdots \binom{\beta_{k-1}}{y_{k-1}}$, and $\widehat{z} = \binom{\gamma_0}{z_0} \cdots \binom{\gamma_{\ell-1}}{z_{\ell-1}}$ the respective colorings.

Next, let $g \coloneqq \mathsf{eval}(x)$, and define the following walks:

$$\begin{aligned}
P_0 &\coloneqq \mathbb{1}H_{\alpha_0} && \to x_0H_{\alpha_1} && \to x_0x_1H_{\alpha_2} && \to \cdots && \to gH_{\alpha_{j-1}}, \\
P_1 &\coloneqq gH_{\beta_0} && \to gy_0H_{\beta_1} && \to gy_0y_1H_{\beta_2} && \to \cdots && \to \mathbb{1}H_{\beta_{k-1}}, \\
P_2 &\coloneqq gH_{\gamma_0} && \to gz_0H_{\gamma_1} && \to gz_0z_1H_{\gamma_2} && \to \cdots && \to \mathbb{1}H_{\gamma_{\ell-1}}.
\end{aligned}$$

Hence, we observe that:

$$\mathcal{L}(\widehat{x}\widehat{y}) = P_0 \circ (gH_{\alpha_{j-1}} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0}), \tag{3.17a}$$

$$\mathcal{L}(\widehat{x}\widehat{z}) = P_0 \circ (gH_{\alpha_{j-1}} \to gH_{\gamma_0}) \circ P_2 \circ (\mathbb{1}H_{\gamma_{\ell-1}} \to \mathbb{1}H_{\alpha_0}), \tag{3.17b}$$

$$g\mathcal{L}(\widehat{y}\widehat{z}^{-1}) = P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0}). \tag{3.17c}$$

(In checking this, the reader is advised that they must sometimes use Remark 3.7. For example, in the $P_0$ piece of $\mathcal{L}(\widehat{x}\widehat{y})$, the tail of the final arrow would be $x_0 \cdots x_{j-2}H_{\alpha_{j-1}}$ per Definition 3.5; however, as noted in Remark 3.7 this is the same as $x_0 \cdots x_{j-2}x_{j-1}H_{\alpha_{j-1}} = \mathsf{eval}(x)H_{\alpha_{j-1}}$.)

Using this, we calculate:

$$\begin{aligned}
\triangle(\mathcal{L}(\widehat{x}\widehat{y})) &= \triangle(P_0 \circ (gH_{\alpha_{j-1}} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0})) && \text{(Equation (3.17a))}\\
&= \triangle(P_0 \circ (gH_{\alpha_{j-1}} \to gH_{\beta_0} \to gH_{\gamma_0} \to gH_{\alpha_{j-1}} \to gH_{\gamma_0} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0})) \\
& && \text{(backtracking)}\\
&\leq \triangle(P_0 \circ (gH_{\alpha_{j-1}} \to gH_{\gamma_0} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0})) + 1 && \text{(subadditivity and Fact 3.15)}\\
&= \triangle(P_0 \circ (gH_{\alpha_{j-1}}, gH_{\gamma_0}) \circ P_2 \circ (\mathbb{1}H_{\gamma_{\ell-1}} \to \mathbb{1}H_{\alpha_0} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0})) + 1 \\
& && \text{(backtracking)}\\
&\leq \triangle((\mathbb{1}H_{\alpha_0} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0}) \circ P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0})) + \triangle(\widehat{x}\widehat{z}) + 1 \\
& && \text{(subadditivity and Equation (3.17b))}\\
&= \triangle(P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0}, \mathbb{1}H_{\alpha_0} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0})) + \triangle(\widehat{x}\widehat{z}) + 1 && \text{(cyclic symmetry)}\\
&= \triangle(P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\alpha_0} \to \mathbb{1}H_{\gamma_{\ell-1}} \to \mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0})) + \triangle(\widehat{x}\widehat{z}) + 1 \\
& && \text{(backtracking)}\\
&\leq \triangle(P_1 \circ (\mathbb{1}H_{\beta_{k-1}} \to \mathbb{1}H_{\gamma_{\ell-1}}) \circ P_2^{-1} \circ (gH_{\gamma_0} \to gH_{\beta_0})) + \triangle(\widehat{x}\widehat{z}) + 2 && \text{(subadditivity and Fact 3.15)}\\
&= \triangle(g\mathcal{L}(\widehat{y}\widehat{z}^{-1})) + \triangle(\widehat{x}\widehat{z}) + 2 && \text{(subadditivity and Equation (3.17c))}\\
&= \triangle(\mathcal{L}(\widehat{y}\widehat{z}^{-1})) + \triangle(\widehat{x}\widehat{z}) + 2, && \text{(translation symmetry)}
\end{aligned}$$

as desired. □

Using Fact 3.13, we immediately conclude:

**Corollary 3.18.** *Let $p, q, u, v$ be words over $\mathcal{H}$ such that $puq$ and $uv^{-1}$, and therefore $pvq$, are relators. Then for all colorings $\widehat{p}, \widehat{q}, \widehat{u}, \widehat{v}$ of $p, q, u, v$ respectively, $\triangle(\widehat{p}\widehat{v}\widehat{q}) \leq \triangle(\widehat{p}\widehat{u}\widehat{q}) + \triangle(\widehat{u}\widehat{v}^{-1}) + 2$.*

Finally, by repeatedly using the above with the "recoloring" words from Fact 3.11, we obtain the following useful lemma:

**Lemma 3.19.** *Let $w$ be a relator over $\mathcal{H}$ and $\widehat{w}$, $\widehat{w}'$ two colorings. Then $\triangle(\widehat{w}') \leq \triangle(\widehat{w}) + 2|w|$.*

## 3.3 Bounding area via derivations on words

Recall our notations from §1.2: $\bigcup \mathcal{H}$ is the union of the subgroups $H_\lambda$ (regarded as a subset of $G$); a *relator* is a word over $\mathcal{H}$ evaluating to $\mathbb{1}$ in $G$; $\mathcal{R}_\ell$ is the set of relators of length at most $\ell$; and the *in-subgroup relators* $\mathcal{R}_\ell^{\text{common}} \subseteq \mathcal{R}_\ell$ consisting of relators wherein all elements are contained in a single common subgroup $H_\lambda$. We are now prepared to define the *fillable relators* which we promised in §1.2 to complete the technical description of our main theorem, Theorem 1.12.

**Definition 3.20** (Fillable relators)**.** Let $G$ be a group, $\mathcal{H} = (H_\lambda < G)_{\lambda \in \Lambda}$ an indexed subgroup family, and $\triangle$ an area function. For integers $\ell \geq 3$, $t \geq 1$, we define the (length-$\leq \ell$) *$t$-fillable relators* $\mathcal{R}_{\ell,t}(\triangle) \subseteq \mathcal{R}_\ell$ to be the set of relators $r = \langle x_0 \rangle \cdots \langle x_{\ell'-1} \rangle$ of length $\ell' \leq \ell$ admitting a coloring $\widehat{r} = \binom{\lambda_0}{x_0}\binom{\lambda_1}{x_1} \cdots \binom{\lambda_{\ell'-1}}{x_{\ell'-1}}$ satisfying $\triangle(\widehat{r}) \leq t$. ◊

Next, we give two simple ways to establish that certain relators are fillable. Firstly, Fact 3.21 below follows immediately from Fact 3.12:

**Fact 3.21.** *For every $\ell \in \mathbb{N}$ (and every area function $\triangle$), $\mathcal{R}_\ell^{\text{common}} \subseteq \mathcal{R}_{\ell,t}(\triangle)$. In words, all in-subgroup relators are 0-fillable. (Usefully, these include length-3 relators of the form $st\mathbb{1}$ where $s, t$ are inverses in $G$.)*

Second, Proposition 3.22 below follows immediately from the fact that *all* loops over a complex with vanishing homology are fillable:

**Proposition 3.22.** *If $\mathfrak{CC}(G;\mathcal{H})$ is homologically 1-connected over $\Gamma$, then for every $\ell \in \mathbb{N}$, we have $\mathcal{R}_\ell \subseteq \mathcal{R}_{\ell,t}(\triangle_{\mathrm{H}}^{\Gamma})$ where $t \coloneqq |(\mathfrak{CC}(G;\mathcal{H}))(2)|$ is the number of triangles in $\mathfrak{CC}(G;\mathcal{H})$. In words, all relators are fillable for sufficiently large $t$.*

**Remark 3.23.** Given $G, \mathcal{H}, \bigcup \mathcal{H}, \mathcal{R}_{\ell,t}(\triangle)$ as above, the reader may find it helpful in what follows to keep in mind the group presentation $\mathcal{P}_{\ell,t} = \langle \bigcup \mathcal{H} \mid \mathcal{R}_{\ell,t}(\triangle) \rangle$, and the associated Dehn function. Strictly speaking, we won't quite refer to these concepts for a few minor technical reasons: we prefer not to have formal inverse symbols; we prefer to think of reducing trivial words to $\mathbb{1}$ rather than to the empty word; and, we prefer to freely allow cyclic shifts and inverses. $\Diamond$

**Remark 3.24.** When studying the Dehn function with respect to a presentation, one typically also considers "free reduction" (replacing $ss^{-1}$ by the empty word, $s \in \bigcup \mathcal{H}$) and its opposite, "free expansion". But these are already included in the preceding formulation via two derivation steps.[15] For example, to implement free reduction on a word of the form $wss^{-1}x$ where at least one of $w, x$ is nonempty — say $x = x_1 \cdots x_m$, $m \geq 1$ — we can first reduce $ss^{-1}$ to $\mathbb{1}$ per Fact 3.21. We can then perform "$\mathbb{1}$-deletion" by reducing $\mathbb{1}x_1$ to $x_1$ (again by Fact 3.21, since "$\mathbb{1}x_1 \equiv x_1$" is "in-subgroup" for any subgroup containing $x_1$). $\Diamond$

Together, these observations give:

**Theorem 3.25.** *Let $w$ be a relator over $\bigcup \mathcal{H}$ and $\widehat{w}$ any coloring. Then for every area function $\triangle$, $\triangle(\widehat{w}) \leq (t + 4\ell) \cdot \mathrm{area}(w; \mathcal{R}_{\ell,t}(\triangle))$.*

*Proof.* Let $\delta = \mathrm{area}(w; \mathcal{R}_{\ell,t}(\triangle)) < \infty$ (else there is nothing to prove) and let $w_0, w_1, \ldots, w_\delta$ denote a sequence of words with $w_0 = \mathbb{1}$ and $w_\delta = w$ in which each $w_i$ is derived from $w_{i-1}$ via some relator in $\mathcal{R}_{\ell,t}(\triangle)$. We make the simple observation that

$$|w_\delta| \leq \ell\delta, \tag{3.26}$$

since any application of an $\mathcal{R}_{\ell,t}(\triangle)$-relator can increase length by at most $\ell$ (in fact, $\ell - 1$). We also note that inductively (on the definition of derivations), each $w_i$ is a relator. Thus for every coloring $\widehat{w}_i$ of $w_i$, $\triangle(\widehat{w}_i)$ is well defined (albeit it might be $\infty$).

Our main goal will be to inductively define colorings $\widehat{w}_0, \widehat{w}_1, \ldots, \widehat{w}_\delta$ of $w_0, \ldots, w_\delta$, respectively, (starting from an arbitrary coloring $\widehat{w}_0$ of $w_0 = \mathbb{1}$) and show that

$$\triangle(\widehat{w}_i) \leq \triangle(\widehat{w}_{i-1}) + t + 2\ell. \tag{3.27}$$

This establishes $\triangle(\widehat{w}_\delta) \leq (t + 2\ell) \cdot \delta$ by induction (note that $\triangle(\widehat{w}_0) = 0$ since $\mathcal{L}(\widehat{w}_0)$ is a self-loop). Then the proof is completed by observing that although the coloring $\widehat{w}$ of $w$ given in the theorem might not equal the final coloring $\widehat{w}_\delta$ we produced, we have

$$\triangle(\widehat{w}) \leq \triangle(\widehat{w}_\delta) + 2|w| \leq \triangle(\widehat{w}_\delta) + 2\ell\delta \tag{3.28}$$

by Lemma 3.19.

To define $\widehat{w}_i$ and establish Equation (3.27), suppose as in Equation (1.10) that

$$w_{i-1} \sim puq, \quad w_i \sim pvq, \quad r \sim uv^{-1} \in \mathcal{R}_{\ell,t}(\triangle), \tag{3.29}$$

and $w_{i-1}$, $w_i$, and $r$ are all relators. By definition of $\mathcal{R}_{\ell,t}(\triangle)$ and Fact 3.13, we may infer that there is a coloring $\widehat{u} = \binom{\gamma_0}{u_0}\binom{\gamma_1}{u_1} \cdots \binom{\gamma_{|u|-1}}{u_{|u|-1}}$ of $u$ and similarly a coloring $\widehat{v}$ of $v$ such that $\widehat{r} := (\widehat{u})(\widehat{v})^{-1}$ has $\triangle(\widehat{r}) \leq t$. Note that the coloring of the $u$-symbols in $\widehat{r}$ might not agree with the coloring of the $u$-symbols in $\widehat{w}_i$. However, if we let $\widehat{r}'$ be a recoloring of $\widehat{r}$ in which the $u$-colors *do* agree with those in $\widehat{w}_{i-1}$, we may conclude from Lemma 3.19 that

$$\triangle(\widehat{r}') \leq \triangle(\widehat{r}) + 2|r| \leq t + 2\ell. \tag{3.30}$$

We may now naturally define the coloring $\widehat{w}_i$ for $w_i$ by using the $p$- and $q$-colors from $\widehat{w}_{i-1}$, the $u$-colors appearing in both $\widehat{w}_{i-1}$ and $\widehat{r}$, and the $v$-colors from $\widehat{r}$. Now finally applying Corollary 3.18 yields Equation (3.27), completing the proof. $\square$

[15] Except in the case where free reduction reduces a word to the empty word. However, we will only ever consider reducing to $\mathbb{1}$.

## 3.4 Back to loops

We now translate Theorem 3.25 into a statement about loops (closed walks) $L$ in the coset complex $\mathfrak{CC}(G;\mathcal{H})$. Using notation from the previous section:

**Fact 3.31** (Loops to words). *Let $L$ be a closed loop in $\mathfrak{CC}(G;\mathcal{H})$ of length $r$. Then there is a word $w$ over $\bigcup\mathcal{H}$ of length $r$ and a coloring $\widehat{w}$ of $w$ such that $\mathcal{L}(\widehat{w}) = L$.*

*Proof.* Write the loop $L$ as $C_0 \to C_1 \to \cdots \to C_{r-1} \to C_0$, where $C_i$ is a coset of $H_{\lambda_i}$ for each $i \in [r]$. Since the coset complex $\mathfrak{CC}(G;\mathcal{H})$ is partite, consecutive vertices in loop $L$ are of different "colors" (i.e., they are not cosets of the same subgroup). We may select elements $g_0, g_1, \ldots, g_{r-1} \in G$ that "witness" each edge in $L$; so $g_i \in C_i \cap C_{i+1}$ (indices taken mod $r$). Now for $x_i \coloneqq g_{i-1}^{-1} g_i$, then $x_i \in H_{\lambda_i}$ (by Fact 2.2); and, $x_0 x_1 \cdots x_{r-1} = \mathbb{1}$. This word $\langle x_0\rangle\langle x_1\rangle \cdots \langle x_{r-1}\rangle$ will be the required $w$. Moreover, it is easy to see that translating $L$ by $g_{r-1}^{-1}$ (in the sense of Fact 2.10) transforms it into the form $\mathcal{L}(\widehat{w})$ for colored word $\widehat{w} = \binom{\lambda_0}{x_0}\binom{\lambda_1}{x_1} \cdots \binom{\lambda_{r-1}}{x_{r-1}}$, which (again) has no two consecutive colored elements of the same color, and also has $\mathsf{eval}(\widehat{w}) = \mathbb{1}$. □

**Corollary 3.32.** *Let $L$ be a closed loop in $\mathfrak{CC}(G;\mathcal{H})$ of length $r$ and $\triangle$ an area function. Then there is a word $w$ over $\bigcup\mathcal{H}$ of length $r$ such that $\triangle(L) \le (t + 4\ell) \cdot \operatorname{area}(w; \mathcal{R}_{\ell,t}(\triangle))$.*

*Proof.* We invoke the prior Fact 3.31 to get a colored word $\widehat{w}$ such that $\mathcal{L}(\widehat{w}) = L$, and apply Theorem 3.25 to this $\widehat{w}$. □

## 3.5 The relative Dehn method

Given all this setup, we can finally prove Theorem 1.12 (giving coboundary expansion bounds in coset complexes). We restate the theorem here:

**Theorem 1.12** (Relative Dehn method). *Let $\Gamma$ be any commutative ring and consider the $d$-dimensional coset complex $\mathfrak{CC}(G;\mathcal{H})$. Suppose that there exist $R_0, \ell, t, \delta \in \mathbb{N}$ such that:*

1. *every element in $G$ can be written as the evaluation of a length-($\le R_0$) word over $\mathcal{H}$.*
2. *every $w \in \mathcal{R}_{2R_0+1}$ satisfies $\operatorname{area}(w; \mathcal{R}_{\ell,t}(\triangle^{\Gamma}_{\mathrm{H}})) \le \delta$.*

*Then $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$ and is $(2R_0+1, O((t+\ell)\delta))$-homologically taut over $\Gamma$, and therefore has $1$-coboundary expansion at least $\Omega(\frac{1}{(t+\ell)\delta})$ over $\Gamma$.*

*Proof.* The first condition (by Proposition 1.6) implies that $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$. For the second condition, let $L$ be a loop of length at most $2R_0 + 1$ in $\mathfrak{CC}(G;\mathcal{H})$. By Corollary 3.32 there exists a word $w$ over $\bigcup\mathcal{H}$ of length at most $2R_0 + 1$ such that $\triangle^{\Gamma}_{\mathrm{H}}(L) \le (t + 4\ell) \cdot \operatorname{area}(w; \mathcal{R}_{\ell,t}(\triangle^{\Gamma}_{\mathrm{H}}))$. By assumption, $\operatorname{area}(w; \mathcal{R}_{\ell,t}(\triangle^{\Gamma}_{\mathrm{H}})) \le \delta$. Hence $\triangle^{\Gamma}_{\mathrm{H}}(L) \le (t+4\ell)\cdot\delta = O(\delta(t+\ell))$. Therefore, $\mathfrak{CC}(G;\mathcal{H})$ is $(2R_0+1, O((t+\ell)\delta))$-homologically taut over $\Gamma$. The conclusion of coboundary expansion from tautness follows by the (homology) cones method (Theorem 2.36). □

## 3.6 Lifting fillings between coset complexes

In this subsection, we give another useful tool for proving that some relators are $t$-fillable in a coset complex, namely, by "lifting" fillable relators from another complex mapping "homomorphically":

**Theorem 3.33.** *Let $G, \bar{G}$ be groups and $f : G \to \bar{G}$ a group homomorphism. Let $\mathcal{H} = (H_\lambda < G)_{\lambda\in\Lambda}$ be a $\Lambda$-indexed subgroup family for $G$, and $\bar{\mathcal{H}} = (\bar{H}_\lambda \subseteq \bar{G})_{\lambda\in\Lambda}$ a $\Lambda$-indexed subgroup family for $\bar{G}$. Further, assume that $f(H_\lambda) \subseteq \bar{H}_\lambda$ for every $\lambda \in \Lambda$. For $\ell, t \in \mathbb{N}$, let $\mathcal{R}_{\ell,t}(\triangle^{\Gamma}_{\mathrm{H}})$ denote the set of $t$-fillable, length-$\le \ell$ relators in $\mathfrak{CC}(G;\mathcal{H})$, and let $\bar{\mathcal{R}}^{\Gamma}_{\ell,t}$ denote the set of $t$-fillable, length-$\le \ell$ relators in $\mathfrak{CC}(\bar{G};\bar{\mathcal{H}})$. Given any word $w = (x_0, \ldots, x_{\ell-1})$ over $\bigcup\mathcal{H}$, define a corresponding word $f(w) \coloneqq (f(x_0), \ldots, f(x_{\ell-1}))$ over $\bigcup\bar{\mathcal{H}}$.*[16]

*Then for $\ell, t \in \mathbb{N}$ and $w \in \mathcal{R}_{\ell,t}(\triangle^{\Gamma}_{\mathrm{H}})$, we have $f(w) \in \bar{\mathcal{R}}^{\Gamma}_{\ell,t}$.*

[16] Note that since $f$ is a homomorphism, if $w$ is a relator, then so is $f(w)$.

In words, the image of a $t$-fillable relator in $\mathfrak{CC}(G;\mathcal{H})$ is $t$-fillable in $\mathfrak{CC}(\bar{G};\bar{\mathcal{H}})$.

*Proof.* Let $w = \langle x_0\rangle \cdots \langle x_{j-1}\rangle$ be a relator assumed to be $t$-fillable for $G$ ($j \le \ell$). This means that there is a coloring $\widehat{w} = \binom{\lambda_0}{x_0}\cdots\binom{\lambda_{j-1}}{x_{j-1}}$ of $w$ (each $x_i \in H_{\lambda_i}$) satisfying $\triangle(\widehat{w}) \le t$. Note that $\widehat{f}(w) \coloneqq \binom{\lambda_0}{f(x_0)}\cdots\binom{\lambda_{j-1}}{f(x_{j-1})}$ is a valid coloring of $f(w)$ (this is because each $f(x_i) \in f(H_{\lambda_i}) \subseteq \bar{H}_{\lambda_i}$). Hence it suffices to show that $\bar{\triangle}^{\Gamma}\widehat{f}(w) \le t$. Write $\mathfrak{X} \coloneqq \mathfrak{CC}(G;\mathcal{H})$ and $\bar{\mathfrak{X}} \coloneqq \mathfrak{CC}(\bar{G};\bar{\mathcal{H}})$ for short.

**Lifting faces.** First, we claim that the homomorphism $f$ *extends to a map of faces* between the complexes $\mathfrak{X}$ and $\bar{\mathfrak{X}}$. For $0 \le i < |\mathcal{H}|$, we define

$$f_i : \mathfrak{X}(i) \to \bar{\mathfrak{X}}(i)$$

in the natural way: $f_i(\{xH_{\lambda_0}, \ldots, xH_{\lambda_i}\}) = \{f_i(x)\bar{H}_{\lambda_0}, \ldots, f_i(x)\bar{H}_{\lambda_i}\}$. To check that this map is well-defined, suppose that $\{xH_{\lambda_0}, \ldots, xH_{\lambda_i}\} = \{x'H_{\lambda_0}, \ldots, x'H_{\lambda_i}\}$. Then for each $0 \le j \le i$, $xH_{\lambda_j} = x'H_{\lambda_j}$, therefore $x^{-1}x' \in H_{\lambda_j}$, therefore $f_i(x)^{-1}f_i(x') = f_i(x^{-1}x') \in f_i(H_{\lambda_j}) \subseteq \bar{H}_{\lambda_j}$ and so $f_i(x)\bar{H}_{\lambda_j} = f_i(x')\bar{H}_{\lambda_j}$.

Having defined $f_i$ on faces, it extends naturally to a map on oriented faces and therefore to $i$-chains $f_i : C^{\Gamma}(\mathfrak{X}; i) \to C^{\Gamma}(\bar{\mathfrak{X}}; i)$. Further, $|\mathrm{supp}(f_i(T))| \le |\mathrm{supp}(T)|$ for all $T \in \mathfrak{X}(i)$.

**Lifting boundaries.** For $0 \le i < |\mathcal{H}|$, let $\partial_i : C^{\Gamma}(\mathfrak{X}; i) \to C^{\Gamma}(\mathfrak{X}; i-1)$ denote the boundary operator for $i$-faces in $\mathfrak{X}$ and similarly for $\bar{\partial}_i$ in $\bar{\mathfrak{X}}$. Our key claim is the following: For every $i < |\mathcal{H}|$,

$$f_{i-1}\partial_i = \bar{\partial}_i f_i. \tag{3.34}$$

This is equivalent to the commutative diagram:

$$\begin{tikzcd}
C^{\Gamma}(\mathfrak{X}; i) \arrow[r, "\partial_i"] \arrow[d, "f_i"'] & C^{\Gamma}(\mathfrak{X}; i-1) \arrow[d, "f_{i-1}"] \\
C^{\Gamma}(\bar{\mathfrak{X}}; i) \arrow[r, "\bar{\partial}_i"'] & C^{\Gamma}(\bar{\mathfrak{X}}; i-1)
\end{tikzcd}$$

It suffices to verify this applied to a 1-sparse chain $g \cdot 1_{(xH_{\lambda_0}, \ldots, xH_{\lambda_i})}$. We have:

$$\begin{tikzcd}
g \cdot 1_{(xH_{\lambda_0}, \ldots, xH_{\lambda_i})} \arrow[r, mapsto, "\partial_i"] \arrow[d, mapsto, "f_i"'] & g \cdot \sum_{j=0}^{i} (-1)^j 1_{(xH_{\lambda_0}, \ldots, xH_{\lambda_i})\setminus j} \arrow[d, mapsto, "f_{i-1}"] \\
g \cdot 1_{(f(x)H_{\lambda_0}, \ldots, f(x)H_{\lambda_i})} \arrow[r, mapsto, "\bar{\partial}_i"'] & g \cdot \sum_{j=0}^{i} (-1)^j 1_{(f(x)\bar{H}_{\lambda_0}, \ldots, f(x)\bar{H}_{\lambda_i})\setminus j}
\end{tikzcd}$$

as desired.

**Lifting fillings.** Recall the colored words $\widehat{w}$ and $\widehat{f}(w)$ and the corresponding closed walks $\mathcal{L}(\widehat{w}), \mathcal{L}(\widehat{f}(w))$:

$$\mathbb{1}H_{\lambda_0} \to x_0H_{\lambda_1} \to \cdots \to x_0 \cdots x_{i-2}H_{\lambda_{i-1}} \to \mathbb{1}H_{\lambda_0}$$
$$\text{and } \mathbb{1}\bar{H}_{\lambda_0} \to f(x_0)\bar{H}_{\lambda_1} \to \cdots \to f(x_0)\cdots f(x_{i-2})\bar{H}_{\lambda_{i-1}} \to \mathbb{1}\bar{H}_{\lambda_0}$$

in $\mathfrak{X}$ and $\bar{\mathfrak{X}}$, respectively. As we see from the preceding paragraph, $f_1$ maps the edges of the walk $\mathcal{L}(\widehat{w})$ to those of the walk $\mathcal{L}(\widehat{f}(w))$ (using that $f$ is a homomorphism). That is, $f_1[\mathcal{L}(\widehat{w})] = [\mathcal{L}(\widehat{f}(w))]$ as 1-chains in $\bar{\mathfrak{X}}$. Now let $T \in C^{\Gamma}(\mathfrak{X}; 2)$ be a $\Gamma$-filling of $[\mathcal{L}(\widehat{w})]$ with $|T| \le t$. Since $\partial_2 T = [\mathcal{L}(\widehat{w})]$, we deduce that

$$\bar{\partial}_2(f_2T) = f_1\partial_2 T = f_1[\mathcal{L}(\widehat{w})] = [\mathcal{L}(\widehat{f}(w))], \tag{3.35}$$

where we used Equation (3.34). Hence $f_2T \in C^{\Gamma}(\bar{\mathfrak{X}}; 2)$ is a $\Gamma$-filling of $[\mathcal{L}(\widehat{f}(w))]$, and its size is at most $t$, as desired. $\square$

# 4 Chevalley coset complexes and their properties

In this section, we turn to describing the specific "Chevalley" coset complexes that our the subject of our main application, Theorem 1.19.

## 4.1 Root systems

To define these coset complexes, we first have to define the underlying groups, called *Chevalley groups*. In turn, to describe these, we need a geometric notion called a *root system*.

An (irreducible) *root system* of rank[17] $d$ is a finite set of vectors $\Phi$ lying in a $d$-dimensional real vector space satisfying certain symmetry conditions. They are completely classified into four infinite families, $(A_d)_{d\geq 1}$, $(B_d)_{d\geq 2}$, $(C_d)_{d\geq 3}$, $(D_d)_{d\geq 4}$, plus five others ($G_2$, $F_4$, $E_6$, $E_7$, $E_8$). We do not state the definition here, as we will not need it; see standard references, e.g., [Hum72, §11-12] or [Hal03, §8].

In turn, root systems are used to help classify finite groups of reflections in $\mathbb{R}^d$. An important concept for root systems is that of a base:

**Definition 4.1.** A *base* for rank-$d$ root system $\Phi$ is a subset $\Pi$ with $|\Pi| = d$ such that $\mathrm{span}_{\mathbb{N}}(\Pi)$ contains one of $\pm\zeta$ for every $\zeta \in \Phi$. Here, $\mathrm{span}_{\mathbb{N}}(P)$ (respectively, $\mathrm{span}_{\mathbb{N}^+}(P)$) denotes all nonnegative (respectively, positive) linear combinations of vectors in $P$. Each root system has a unique base up to isometry, and every base is a basis for the underlying $d$-dimensional vector space. ◊

**Definition 4.2.** Suppose $I \subseteq \Phi$ is linearly independent and $\zeta \in \mathrm{span}_{\mathbb{N}}(I)$; say $\zeta = \sum_{\eta\in I} c_\eta \eta$. Then we write $\mathrm{height}_I(\zeta) \coloneqq \sum_{\eta\in I} c_\eta$ for the *height* of $\zeta$ with respect to $P$. ◊

The root systems relevant for our work are the rank-3 ones, namely $A_3$, $B_3$, and $C_3$. In fact, since we are leaving $C_3$ for later work, we only define here $A_3$ and $B_3$.[18]

***A*-type.** The $A_d$-type root system $\Phi_{A_d}$ consists of vectors in $\mathbb{R}^{d+1}$ of the form $\mathbf{e}_i - \mathbf{e}_j$, $i \neq j \in [d]$, where $\mathbf{e}_i$ denotes the $i$-th standard basis vector. (This is rank $d$, since all vectors are orthogonal to $(1, 1, \dots, 1)$.) One base for $A_3$, depicted in Figure 4 below, is

$$\Pi_{A_3} = \{\alpha, \beta, \gamma\}, \text{where } \alpha = (1, -1, 0, 0),\ \beta = (0, 1, -1, 0),\ \gamma = (0, 0, 1, -1); \tag{4.3}$$

we also name the root vector

$$\delta = -(\alpha + \beta + \gamma) = (-1, 0, 0, 1). \tag{4.4}$$

The nonnegative span of these roots is:

$$\Phi^+_{A_3} \coloneqq \mathrm{span}_{\mathbb{N}}(\{\alpha, \beta, \gamma\}) \cap \Phi_{A_3} = \{\alpha, \beta, \gamma, \alpha + \beta, \alpha + \beta, \alpha + \beta + \gamma\}.$$

The corresponding height function is

$$\begin{aligned}
&\mathrm{height}_{A_3}(\alpha) = \mathrm{height}_{A_3}(\beta) = \mathrm{height}_{A_3}(\gamma) = 1,\\
&\mathrm{height}_{A_3}(\alpha + \beta) = \mathrm{height}_{A_3}(\beta + \gamma) = 2,\\
&\mathrm{height}_{A_3}(\alpha + \beta + \gamma) = 3.
\end{aligned}$$

***B*-type.** The type-$B_d$ root system $\Phi_{B_d}$ consists of all integer vectors in $\mathbb{R}^d$ of length 1 ("short roots") or $\sqrt{2}$ ("long roots"). One canonical base for $B_3$, depicted in Figure 5 below, is

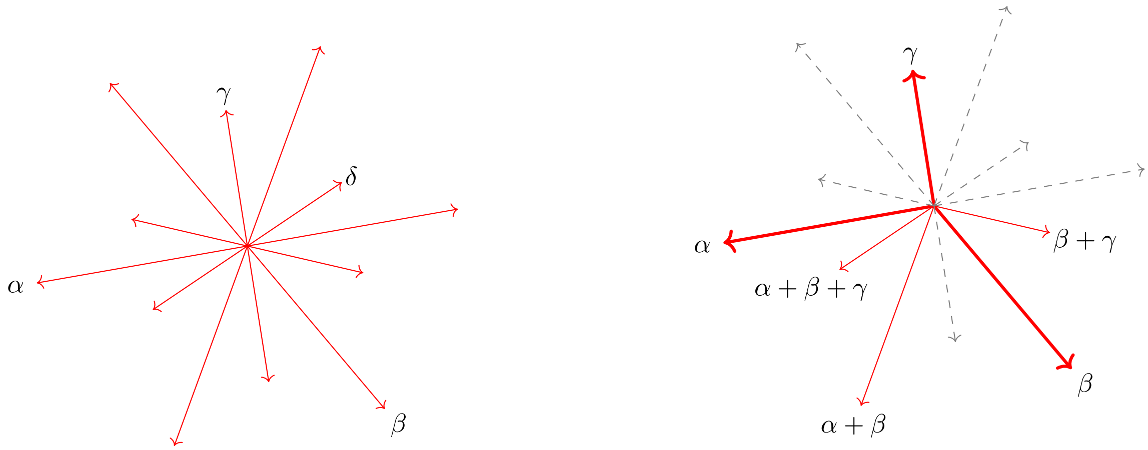
$$\Pi_{B_3} = \{\alpha, \beta, \psi\}, \text{where } \alpha = (1, -1, 0),\ \beta = (0, 1, -1),\ \psi = (0, 0, 1). \tag{4.5}$$

(Note also our slight abuse of notation: $\alpha$, $\beta$ denote formally different vectors in $A_3$ vs. $B_3$.) We will also name the root vector

$$\omega = -(\alpha + \beta + \psi) = (-1, 0, 0). \tag{4.6}$$

[17]We will henceforth only consider irreducible root systems of rank at least 2.

[18]We need not consider $D_3$, because $D_3 = A_3$.




(a) The root system $A_3 \subseteq \mathbb{R}^4$ under an orthogonal projection into $\mathbb{R}^3$. We also label the our canonical base roots $\alpha$, $\beta$, $\gamma$, and their negative sum $\delta = -(\alpha+\beta+\gamma)$.

(b) The "link of $\delta$" in $A_3$: $\alpha$, $\beta$, and $\gamma$ are drawn as solid lines; all roots which are nonnegative integer combinations of these are drawn as colored thin lines; and the remaining roots are dashed and gray.

Figure 4: The root system $A_3$ and a specific "link" within it.

Note that $\alpha$ and $\beta$ are "long" and $\psi$ and $\omega$ are "short". We have

$$\Phi^{\mathrm{sm},+}_{B_3} \coloneqq \mathrm{span}_{\mathbb{N}}(\{\beta,\psi,\omega\}) \cap \Phi_{B_3} = \{\beta,\omega,\psi,\beta+\omega,\omega+\psi,\beta+2\psi,\beta+\psi+\omega\},$$
$$\Phi^{\mathrm{lg},+}_{B_3} \coloneqq \mathrm{span}_{\mathbb{N}}(\{\alpha,\beta,\psi\}) \cap \Phi_{B_3} = \{\alpha,\beta,\psi,\alpha+\beta,\beta+\psi,\beta+2\psi,\alpha+\beta+\psi,\alpha+\beta+2\psi,\alpha+2\beta+2\psi\}.$$

The corresponding height functions are

$$\begin{aligned}
&\mathrm{height}^{\mathrm{sm}}_{B_3}(\beta) = \mathrm{height}^{\mathrm{sm}}_{B_3}(\psi) = \mathrm{height}^{\mathrm{sm}}_{B_3}(\omega) = 1,\\
&\mathrm{height}^{\mathrm{sm}}_{B_3}(\beta+\omega) = \mathrm{height}^{\mathrm{sm}}_{B_3}(\omega+\psi) = 2,\\
&\mathrm{height}^{\mathrm{sm}}_{B_3}(\beta+2\psi) = \mathrm{height}^{\mathrm{sm}}_{B_3}(\beta+\psi+\omega) = 3,
\end{aligned}$$

and

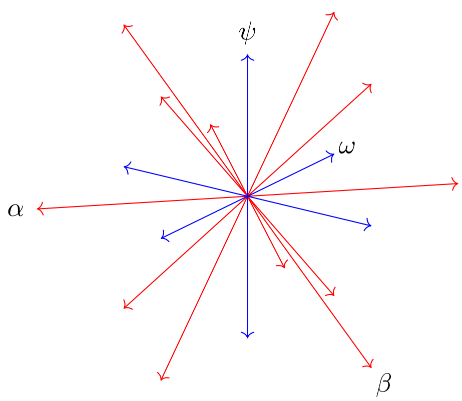
$$\begin{aligned}
&\mathrm{height}^{\mathrm{lg}}_{B_3}(\alpha) = \mathrm{height}^{\mathrm{lg}}_{B_3}(\beta) = \mathrm{height}^{\mathrm{lg}}_{B_3}(\psi) = 1,\\
&\mathrm{height}^{\mathrm{lg}}_{B_3}(\alpha+\beta) = \mathrm{height}^{\mathrm{lg}}_{B_3}(\beta+\psi) = 2,\\
&\mathrm{height}^{\mathrm{lg}}_{B_3}(\beta+2\psi) = \mathrm{height}^{\mathrm{lg}}_{B_3}(\alpha+\beta+\psi) = 3,\\
&\mathrm{height}^{\mathrm{lg}}_{B_3}(\alpha+\beta+2\psi) = 4,\\
&\mathrm{height}^{\mathrm{lg}}_{B_3}(\alpha+2\beta+2\psi) = 5.
\end{aligned}$$

## 4.2 Chevalley groups

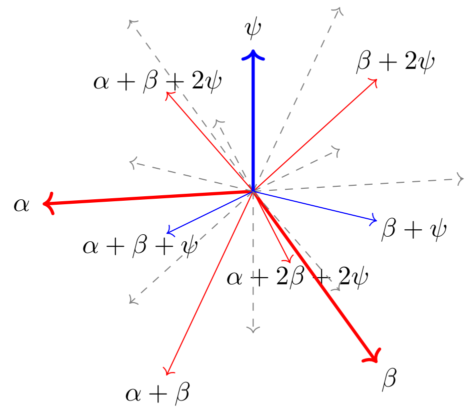
Each root system can be combined with (almost) any finite field $\mathbb{F}$ to produce a finite simple (or nearly-simple) group called a *(universal) Chevalley group*. We define these groups abstractly via their *Steinberg presentation*:

**Definition 4.7** (Steinberg presentation of a Chevalley group)**.** Let $\Phi$ be a root system and $\mathbb{F}$ a finite field. The corresponding (universal) Chevalley group $G_\Phi(\mathbb{F})$ is generated by elements/symbols[19] of the form $\{\{\zeta,t\}\}$ for $\zeta \in \Phi$ and $t \in \mathbb{F}$. We sometimes call $\zeta$ the "type" of the element and $t$ its "entry". The elements are subject to the following three families of relations:

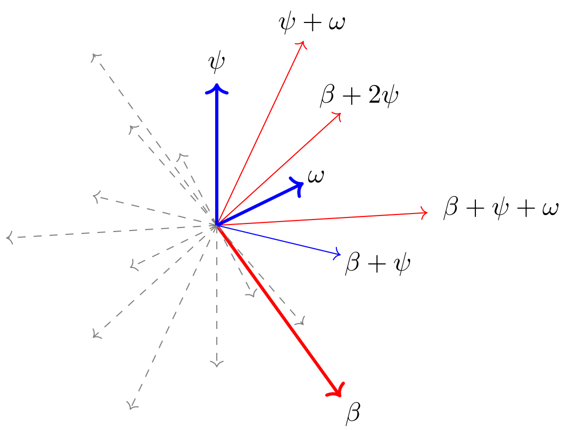
[19] These elements are traditionally written $x_\zeta(t)$, but we will find our subscript-free notation more readable.




(a) The root system $B_3 \subseteq \mathbb{R}^3$. We distinguish "long" and "short" roots as blue and red, respectively. We also label our canonical base roots $\alpha$, $\beta$, $\psi$, and their negative sum $\omega = -(\alpha + \beta + \psi)$.




(b) The "link of $\omega$" in $B_3$: $\alpha$ (long), $\beta$ (long), and $\psi$ (short) are drawn as solid lines; all roots which are nonnegative integer combinations of these are drawn as colored thin lines; and the remaining roots are dashed and gray.




(c) The "link of $\alpha$" in $B_3$: $\beta$ (long), $\psi$ (short), and $\omega$ (short) are drawn as thick lines; all roots which are nonnegative integer combinations of these are drawn as colored thin lines; and the remaining roots are dashed and gray.

Figure 5: The root system $B_3$ and two specific "links" within it.

- "Linearity": For every $\zeta \in \Phi$ and $t, u \in \mathbb{F}$,

$$\{\{\zeta, t\}\}\{\{\zeta, u\}\} = \{\{\zeta, t+u\}\}. \tag{4.8}$$

(These relations imply that $\{\{\zeta, 0\}\} = \mathbb{1}$ and $\{\{\zeta, t\}\}^{-1} = \{\{\zeta, -t\}\}$.)

- "Commutator": For every $\zeta \neq -\eta \in \Phi$ and $t, u \in \mathbb{F}$,

$$[\{\{\zeta, t\}\}, \{\{\eta, u\}\}] = \prod_{\substack{a\zeta+b\eta\in\Phi \\ a,b\in\mathbb{N}^+}} \{\{a\zeta + b\eta, C^{\zeta,\eta}_{a,b} \cdot t^a u^b\}\}, \tag{4.9}$$

where the product is ordered according to a fixed, global total order on $\Phi$, and where $C^{\zeta,\eta}_{a,b} \in \{\pm 1, \pm 2, \pm 3\}$ are certain so-called *Chevalley constants* whose values (depending only on the root ordering, not on $t, u$) we will discuss below.

- "Diagonal": For every $\zeta \in \Phi$ and $t, u \neq 0 \in \mathbb{F}$, $h_\zeta(t)h_\zeta(u) = h_\zeta(tu)$, where $h_\zeta(t) := g_\zeta(t)g_\zeta(-1)$ and $g_\zeta(t) := \{\{\zeta, t\}\}\{\{-\zeta, -t^{-1}\}\}\{\{\zeta, t\}\}$.

In words: the linearity relations state that multiplying elements of the same type adds their entries; the commutator relations say that commuting $\zeta$- and $\eta$-type elements produces a product of $\theta$-type elements over all $\theta$ in $\mathrm{span}_{\mathbb{N}^+}(\{\zeta, \eta\}) \cap \Phi$. ◊

**Remark 4.10.** We give short shrift to the "diagonal" relations, as they will not be very relevant for our work. This is because we mainly study "unipotent" subgroups of $G_\Phi(\mathbb{F})$, which never simultaneously contain elements of both types $\pm\zeta$.

Note also that the "linearity" and "commutator" relations would also make sense if the entries came from a ring (without division), rather than a field. ◊

## 4.3 Graded unipotent subgroups

Before defining the Chevalley coset complexes, it will be helpful to define and study a generalization of the unipotent subgroups of Chevalley groups, where we restrict what entries are allowed for the generators. These subgroups were introduced in the $A_n$-type case in [KO18], and for general Chevalley groups in [OP22]. Throughout this subsection, $\Phi$ denotes a root system.

**Definition 4.11.** Let $I \subset \Phi$ be linearly independent and let $\mathbb{F}_q$ be a finite field. We define the *ungraded unipotent subgroup* $U_I(\mathbb{F}_q) < G_\Phi(\mathbb{F}_q)$ as the subgroup generated by all elements $\{\{\zeta, t\}\}$ where $\zeta \in I$ and $t \in \mathbb{F}_q$. (We omit $\Phi$ from the notation; it will always be clear from context.) Note that for every $J \subseteq I$, we have $U_J(\mathbb{F}_q) \subseteq U_I(\mathbb{F}_q)$; hence $U_I(\mathbb{F}_q)$ admits an $I$-indexed subgroup family $(U_{I\setminus\{i\}}(\mathbb{F}_q))_{i\in I}$. ◊

**Remark 4.12.** In the preceding definition, when $I$ is a *base* for $\Phi$, the resulting $U_I(\mathbb{F}_q)$ is usually termed *the* unipotent subgroup of $G_\Phi(\mathbb{F}_q)$. (Different bases lead to isomorphic subgroups.) As an example, *the* unipotent subgroup of $G_{A_d}(\mathbb{F}_q)$, namely $U_I(\mathbb{F}_q)$ with $I = \{\mathbf{e}_i - \mathbf{e}_{i+1} : 0 \leq i < d\}$, is isomorphic to the group of upper-triangular matrices in $\mathbb{F}^{(d+1)\times(d+1)}$ with 1's on the diagonal. But we will also be interested in $U_I(\mathbb{F}_q)$ in the case where $I$ is linearly independent but not a base. ◊

**Notation 4.13.** Let us introduce some shorthands for the ungraded unipotent groups we are interested in, referring back to §4.1. In the $A_3$ case, all the ungraded unipotent groups are isomorphic, and we will write $U_{A_3}(\mathbb{F}_q)$ for the case of $I = \{\alpha, \beta, \gamma\}$. In the $B_3$ case, there are two subgroups of interest up to isomorphism. We will write $U^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$ for $I = \{\beta, \psi, \omega\}$, i.e., the case of deleting a long root. We will write $U^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$ for $I = \{\alpha, \beta, \psi\}$, i.e., the case of deleting a short root. See Appendix D.1 below for explicit definitions of these three groups via their Steinberg presentations. Each of these groups admits corresponding size-3 indexed subgroup families; each subgroup corresponds to omitting one additional root from $I$ (cf. Definition 4.11). ◊

The following related notion of unipotent subgroup was first studied by [KO18] in the $\Phi = A_d$ case, and by [OP22] in the general $\Phi$ case.

**Definition 4.14.** Let $I \subset \Phi$ be linearly independent, let $\mathbb{F}_q$ be a finite field, and let $x$ be an indeterminate. We define a corresponding *graded unipotent subgroup* $GU_I(\mathbb{F}_q) < G_\Phi(\mathbb{F}_q[x])$ as the subgroup generated by all elements $\{\{\zeta, f\}\}$ where $\zeta \in I$ and $f \in \mathbb{F}_q[x]$ with $\deg(f) \leq \mathrm{height}_I(\zeta)$. (Again, this notation $U_I(\mathbb{F}_q)$ does not show dependence on $\Phi$, but it will always be clear from context. ) Note that for every $J \subseteq I$, we have $GU_J(\mathbb{F}_q) < GU_I(\mathbb{F}_q)$; hence $GU_I(\mathbb{F}_q)$ admits an $I$-indexed subgroup family $(GU_{I\setminus\{i\}}(\mathbb{F}_q))_{i\in I}$. ◊

**Notation 4.15.** Referring back to the ungraded groups defined in Notation 4.13, we write $GU_{A_3}(\mathbb{F}_q)$, $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$, and $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$ for the corresponding *graded* unipotent groups, each of which (again) admits an indexed subgroup family of size 3. ◊

The following proposition characterizes the structure of all elements in the group $GU_I(\mathbb{F}_q)$, and in particular implies that this group is finite:

**Proposition 4.16** ([OP22, Prop. 3.14])**.** *There exists an ordering $\preceq$ on $\operatorname{span}_{\mathbb{N}}(I) \cap \Phi$ s.t. the elements of $GU_I(\mathbb{F}_q)$ can be written uniquely as*

$$\prod_{\zeta \in \operatorname{span}_{\mathbb{N}}(I)\cap\Phi} \{\{\zeta, f_\zeta\}\}, \tag{4.17}$$

*where the product is taken in the $\prec$ ordering, and where $\deg(f_\zeta) \leq \operatorname{height}_I(\zeta)$.*

**Remark 4.18.** A similar (and simpler) analogue of Proposition 4.16 holds for the group $U_I(\mathbb{F}_q)$; the elements in the product are of the form $\{\{\zeta, t\}\}$ where $t \in \mathbb{F}_q$. We do not state it formally as we do not need it. The same holds for Theorem 4.19 and Proposition 4.20 below. ◊

We also have the following very important structure theorem. The theorem follows from [OP22, Prop. 3.14], a key component of which is [Ste16, Lem. 17]; we slightly strengthen it to give quantitative details on *how* the presentation for $GU_I(\mathbb{F}_q)$ derives words:

**Theorem 4.19** (Polynomial presentation)**.** *Let $q$ be an odd prime power, and let $I \subset \Phi \neq G_2$ be linearly independent. There is a function*[20] *$\kappa : \mathbb{N} \to \mathbb{N}$ depending* only *on $I, \Phi$ — and not on $q$ — such that every trivial word of length at most $r$ in the symbols $\{\{\zeta, f\}\}$ ($\zeta \in \operatorname{span}_{\mathbb{N}}(I) \cap \Phi$, $f \in \mathbb{F}_q[x]$, $\deg(f) \leq \operatorname{height}_I(\zeta)$) can be reduced to $\mathbb{1}$ by the use of at most $\kappa(r)$ applications of the linearity relations (Equation (4.8)) and the commutator relations (Equation (4.9)).*

(Similar facts are true for the ungraded groups, but we will not need them in this paper.)

*Proof.* This follows simply by inspecting the proof of [OP22, Prop. 3.14] (which in turn follows Carter's proof [Car89, Thm. 5.3.3] of [Ste16, Lem. 17]). To briefly sketch the proof, given any trivial word $w$ of length $r$ over the symbols $\{\{\zeta, f(x)\}\}$, one repeatedly finds consecutive pairs of symbols that are misordered with respect to $\prec$, and reorders them using the commutator relation. This may increase the length of the word by $O(1)$, but decreases the number of misorderings by 1. One repeats this ($O(r \log r)$ times) until the word's symbols are ordered. Then one repeatedly uses linearity relations ($O(r \log r)$ times in total) to merge consecutive symbols of the same type. The final result is a word of the form in Equation (4.17). But now all the symbols in this word must have entry 0, as the word itself evaluates to $\mathbb{1}$, and Theorem 4.19's uniqueness statement implies $\prod_{\zeta\in\operatorname{span}_{\mathbb{N}}(I)\cap\Phi} \{\{\zeta, 0\}\}$ is the unique representation of $\mathbb{1}$. □

We also have the following useful proposition:

**Proposition 4.20.** *For every $\Phi$, there exists an absolute constant $r_0 \in \mathbb{N}$ (depending only on $\Phi$, but not $q$) s.t. every element $\{\{\zeta, f\}\}$ in $GU_I(\mathbb{F}_q)$ (for $\zeta \in \operatorname{span}_{\mathbb{N}}(I) \cap \Phi$ and $f \in \mathbb{F}_q[x]$, $\deg(f) \leq \operatorname{height}_I(\zeta)$) is the evaluation of a length-($\leq r_0$) word in the symbols $\{\{\eta, t\}\}$ where $\eta \in I$ and $t \in \mathbb{F}_q[x]$ with $\deg(t) \leq 1$.*

In particular, the symbols $\{\{\eta, t\}\}$ with $\eta \in I$ and $t \in \mathbb{F}_q[x]$, $\deg(t) \leq 1$ generate $GU_I(\mathbb{F}_q)$.

*Proof.* By inspection of the bounded generation condition in the proof of [OP22, Lem. 3.13], which uses a number of elements that depends only at worst on the number of roots in $\Phi$ and the maximum "height" of any root in $\Phi$, all of which are bounded by a function only of $\Phi$. □

These propositions can be used to give $m$- and $p$-independent diameter bounds for the corresponding coset complexes.

We now make two further important modifications to this presentation:

[20]The function $\kappa$ has growth rate $\kappa(r) = O(r \log r)$, though we won't need this.

**Definition 4.21.** For the presentation discussed in Theorem 4.19, let us introduce the symbol

$$(\!(\zeta, t, i)\!) := \{\{\zeta, tx^i\}\}.$$

Following [KO21], we call these the *pure-degree* elements. Observe that the linearity and commutator relations that have only pure-degree symbols on the left-hand side also only have pure-degree symbols on the right-hand side. We will term this subset of relations the *pure-degree Steinberg relations.* $\diamondsuit$

By linearity (Equation (4.8)), every element in $GU_I(\mathbb{F}_q[x])$ of the form $\{\{\zeta, f\}\}$ is a product of elements of the form $(\!(\zeta, t, i)\!)$. This gives, together with Theorem 4.19, the following:

**Theorem 4.22** (Pure-degree presentation)**.** *Let $q$ be an odd prime power and let $I \subset \Phi \neq G_2$ be linearly independent.*

- Generation: *$GU_I(\mathbb{F}_q[x])$ is generated by the pure-degree symbols $(\!(\zeta, t, i)\!)$ where $\zeta \in I$, $t \in \mathbb{F}_q$, and $i \in [\mathrm{height}_I(\zeta)]$.*
- Efficient presentation: *There is a function $\kappa : \mathbb{N} \to \mathbb{N}$ depending* only *on $\Phi$ and $I$ — and not on $q$ — such that the following holds. Every relator of length at most $r$ in the pure-degree symbols can be reduced to $\mathbb{1}$ by the use of at most $\kappa(r)$ applications of pure-degree linearity relations (Equation (4.8)) and commutator relations (Equation (4.9)).*

## 4.4 The Chevalley coset complexes

Kaufman and Oppenheim [KO18] showed how to construct bounded-degree HDXs from coset complexes over $G_{A_d}(\mathbb{F}) \cong \mathsf{SL}_{d+1}(\mathbb{F})$, and O'Donnell and Pratt [OP22] generalized their work to all Chevalley groups. (See [GV25] for further generalizations.) Let us recap the main construction and theorem from [OP22]:

**Definition 4.23** ([OP22])**.** Let $\mathbb{F}_q$ be a finite field of characteristic at least 3,[21] and let $\mathbb{F}_{q^m} \cong \mathbb{F}_q[x]/(p(x))$, where $p(x)$ is an irreducible of degree $m$. Consider the Chevalley group $G_\Phi(\mathbb{F}_{q^m})$ (Definition 4.7). Let $\Phi$ be a rank-$d$ root system, and $\Pi \subset \Phi$ a base. Define the "special" root set $\Lambda := \Pi \cup \{\zeta\}$, where $\zeta := -\sum_{\eta\in\Pi} \eta \in \Phi$. For every $I \subsetneq \Lambda$, define the subgroup $\widetilde{U}_I(\mathbb{F}_q, m) < G_\Phi(F_{q^m})$ as the subgroup generated by elements $\{\{\zeta, f\}\}$ for $\zeta \in I$ and $f \in \mathbb{F}_q[x]/(p(x))$ with $\deg(f) \leq \mathrm{height}_I(\zeta)$.[22] Then, we define the $d$-dimensional coset complex:

$$\mathfrak{K}\Phi_q(m) := \mathfrak{CC}\Big(G_\Phi(\mathbb{F}_{q^m}); (\widetilde{U}_{\Lambda\setminus\{\eta\}}(\mathbb{F}_q, m))_{\eta\in\Lambda}\Big). \tag{4.24}$$

$\diamondsuit$

**Notation 4.25.** We use the notations $(\mathfrak{A}^3_q(m))_m$ and $(\mathfrak{B}^3_q(m))_m$ for the coset complex families (in the sense of Definition 4.23) corresponding to the $A_3$ and $B_3$ root systems, respectively. The former were constructed in [KO18] and the latter in [OP22]. $\diamondsuit$

We will need to record a couple of facts about these complexes, before recalling the main theorem about them.

**Proposition 4.26** ([OP22, Thm. 3.18])**.** *In $\mathfrak{K}\Phi_q(m)$, suppose vertex $\sigma$ is a coset of $H_\eta$, $\eta \in \Lambda$. Then the link of $\sigma$ in $\mathfrak{K}\Phi_q(m)$ is isomorphic to the coset complex $\mathfrak{CC}\Big(\widetilde{U}_{\Lambda\setminus\{\eta\}}(\mathbb{F}_q, m); (\widetilde{U}_{\{\eta,\zeta\},m})_{\zeta\in\Lambda\setminus\{\eta\}}\Big)$.*

**Proposition 4.27** ([OP22, Cor. 3.19])**.** *For every $\Phi$, $\mathfrak{K}\Phi_q(m)$ is 0-connected, and further, for every $0 \leq i \leq d-2$, the link of every $i$-face is 0-connected.*

Also, the following follows immediately from the definitions:

**Fact 4.28.** *If $m \geq \max\{\mathrm{height}_I(\zeta) : \zeta \in \mathrm{span}_{\mathbb{N}}(I) \cap \Phi\}$, then every $\widetilde{U}_I(\mathbb{F}_q, m) \cong GU_I(\mathbb{F}_q)$. If $m = 0$, then every $\widetilde{U}_I(\mathbb{F}_q, 0) \cong U_I(\mathbb{F}_q)$.*

This fact, together with Proposition 4.26, motivates the following definition.

[21] See Footnote 6.

[22] Note that since $\Pi$ is linearly independent, every proper subset of $\Lambda$ is also linearly independent.

**Notation 4.29.** We use the notations $\mathfrak{GLA}_3(\mathbb{F}_q)$, $\mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q)$, and $\mathfrak{GLB}_3^{\mathrm{lg}}(\mathbb{F}_q)$, for the coset complexes for the graded unipotent groups $GU_{A_3}(\mathbb{F}_q)$, $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_q)$, and $GU_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$, respectively. (These are each 2-dimensional coset complexes defined using the natural indexed subgroup family, cf. Definition 4.14.) For $m \geq 4$, in $\mathfrak{A}_q^3(m)$, all vertex-links are isomorphic to $\mathfrak{GLA}_3(\mathbb{F}_q)$. For $m \geq 6$, in $\mathfrak{B}_q^3(m)$, all vertex-links are isomorphic to $\mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ or $\mathfrak{GLB}_3^{\mathrm{lg}}(\mathbb{F}_q)$.

We also use the notations $\mathfrak{LA}_3(\mathbb{F}_q)$, $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_q)$, and $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_q)$ to denote the coset complexes for the unipotent groups $U_{A_3}(\mathbb{F}_q)$, $U_{B_3}^{\mathrm{sm}}(\mathbb{F}_q)$, and $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$, respectively, vis-à-vis the subgroups obtained by omitted one additional root (cf. Definition 4.11). Vertex-links in $\mathfrak{A}_q^3(0)$ are isomorphic to $\mathfrak{LA}_3(\mathbb{F}_q)$ and vertex-links in $\mathfrak{B}_q^3(0)$ are isomorphic to $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ or $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_q)$. $\Diamond$

### 4.5 Spectral expansion of Chevalley coset complexes

We now give the main theorem from [OP22] about these coset complexes (excluding the case "$G_2$" for simplicity; see [Pra23] for its treatment).

**Theorem 4.30** ([OP22, Thm. 3.6, Cor. 3.7])**.** *Let $\Phi \neq G_2$ be of rank $d$. Then $(\mathfrak{K}\Phi_q(m))_{m\in N}$ is a strongly explicit family of simplicial complexes on $q^{\Theta(m)}$ vertices, of bounded degree $D = q^{O(1)}$, and has $j$-spectral expansion parameter at most $\frac{1}{\sqrt{q/2-(d-1-j)}}$.*

## 5 Proof of Theorem 1.19

In this section, we discuss the proof of our main application theorem, Theorem 1.19, which we restate here:

**Theorem 1.19** (Main application)**.** *If $q$ is a sufficiently large power of 5, the 2-dimensional simplicial complexes $\widehat{\mathfrak{B}}_q^3(m)$ have 1-cosystolic expansion at least $(\epsilon_0, \mu_0)$ over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, where $\epsilon_0, \mu_0 > 0$ are universal constants.*

### 5.1 Lifting between unipotent subgroups

To prove Theorem 1.19, we develop a final useful notion: homomorphisms between $U_I(\mathbb{F})$ and $GU_I(\widetilde{\mathbb{F}})$ where $\widetilde{\mathbb{F}} \supset \mathbb{F}$ is a field extension. We will use these homomorphisms to "lift" (computer-generated) "fillings" of loops between the corresponding coset complexes. These homomorphisms were implicitly studied in a slightly different context in the $A_3$ case in [KO21, Lemma 7.13].

**Theorem 5.1.** *Let $I \subset \Phi$ be linearly independent, let $\mathbb{F}$ be a finite field, and $\widetilde{\mathbb{F}} \supseteq \mathbb{F}$ a field extension. Consider the two unipotent subgroups $U_I(\mathbb{F})$ and $GU_I(\widetilde{\mathbb{F}})$. Suppose we have field elements $t_{\zeta,b} \in \widetilde{\mathbb{F}}$ for $\zeta \in I$, $b \in \{0,1\}$.*

*Define a map $f : U_I(\mathbb{F}) \to GU_I(\widetilde{\mathbb{F}})$ by specifying that for $\eta = \sum_{\zeta\in I} c_\zeta \zeta$, and $u \in \mathbb{F}$,*

$$f\{\{\eta, u\}\} \coloneqq \{\{\eta, u \textstyle\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{c_\zeta}\}\}.$$

*Then this map is a homomorphism.*

*Proof.* It suffices to check that the $f$-image of every relation in the Steinberg presentation of $U_I(\mathbb{F})$ (the "$m = 1$ case" of Theorem 4.19) is true within $GU_I(\widetilde{\mathbb{F}})$ (the general-$m$ of Theorem 4.19). (This is sometimes called van Dyck's Theorem. In fact, each $f$-image will be a Steinberg relation in the latter group.) It is easy to verify this for the linearity relations, so it remains to verify it for the commutation relations.

In $U_I(\mathbb{F})$, commutation relations are of the form

$$[\{\{\eta, t\}\}, \{\{\theta, u\}\}] = \prod_{\substack{a\eta+b\theta\in\Phi\\ a,b\in\mathbb{N}^+}} \{\{a\eta + b\theta, C^{\eta,\theta} \cdot t^a u^b\}\}.$$

Suppose the roots $\eta$ and $\theta$ have expansions $\eta = \sum_{\zeta\in I} c_\zeta \zeta$ and $\theta = \sum_{\zeta\in I} d_\zeta \zeta$. Then $a\eta+b\theta = \sum_{\zeta\in I}(ac_\zeta+bd_\zeta)$. So by our definition of $f$,

$$f\{\{\eta, t\}\} = \{\{\eta, t \textstyle\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{c_\zeta}\}\},$$

$$f\{\{\theta, u\}\} = \{\{\theta, u\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{d_\zeta}\}\},$$
$$f\{\{a\eta + b\theta, C^{\eta,\theta}\cdot t^a u^b\}\} = \{\{a\eta + b\theta, C^{\eta,\theta}\cdot t^a u^b\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{ac_\zeta + bd_\zeta}\}\}.$$

Applying the commutator relation in $U_I(\widetilde{\mathbb{F}}[x]_{\le 1})$ gives:

$$\begin{aligned}
f\left[\{\{\eta, t\}\}, \{\{\theta, u\}\}\right] &= \left[\{\{\eta, t\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{c_\zeta}\}\}, \{\{\theta, u\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{d_\zeta}\}\}\right] \\
&= \prod_{\substack{a\eta+b\theta\in\Phi \\ a,b\in\mathbb{N}^+}} \{\{a\eta + b\theta, C^{\eta,\theta}\cdot\left(t\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{c_\zeta}\right)^a\left(u\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{d_\zeta}\right)^b\}\} \\
&= \prod_{\substack{a\eta+b\theta\in\Phi \\ a,b\in\mathbb{N}^+}} \{\{a\eta + b\theta, C^{\eta,\theta}t^a u^b\cdot\prod_{\zeta\in I}(t_{\zeta,1}x + t_{\zeta,0})^{ac_\zeta + bd_\zeta}\}\} \\
&= \prod_{\substack{a\eta+b\theta\in\Phi \\ a,b\in\mathbb{N}^+}} f\{\{a\eta + b\theta, C^{\eta,\theta}\cdot t^a u^b\}\},
\end{aligned}$$

and so the image of a commutator relation is indeed a (commutator) relation. □

See Remark D.9 below for an explicit example of these lifting homomorphisms in the $A_3$ and $B_3$ cases.

Note that using the pure degree symbols, the image of an element $\{\{\eta, u\}\}$ in Theorem 5.1 is the multinomial expansion:

$$\prod_{b_\zeta\in[1]:\zeta\in I} (\!(\eta, u\prod_{\zeta\in I} t_{\zeta,b_\zeta}^{c_\zeta}, \sum_{\zeta\in I} c_\zeta b_\zeta)\!). \tag{5.2}$$

We call this general form a *nonhomogeneous lift*. One useful special case that simplifies the notation considerably is the *homogeneous lift*, where for every $\zeta$, either $t_{\zeta,1}$ or $t_{\zeta,0}$ is zero:

**Corollary 5.3** (Homogeneous lifting)**.** *Let $I\subset\Phi$ be linearly independent, let $\mathbb{F}$ be a finite field, and $\widetilde{\mathbb{F}}\supseteq\mathbb{F}$ a field extension. Consider the two unipotent subgroups $U_I(\mathbb{F})$ and $GU_I(\widetilde{\mathbb{F}})$. Suppose we have $(t_\zeta\in\widetilde{\mathbb{F}})_{\zeta\in I}$ and $(b_\zeta\in[1])_{\zeta\in I}$. There is a (unique)* lift *homomorphism $f : U_I(\mathbb{F})\to GU_I(\widetilde{\mathbb{F}})$ such that for every $\zeta\in I$ and $u\in\mathbb{F}$,*

$$f\{\{\zeta, u\}\} = (\!(\zeta, ut_\zeta, b_\zeta)\!).$$

*The image of an element of type $\eta = \sum_{\zeta\in I} c_\zeta\zeta$ is:*

$$f\{\{\eta, u\}\} = (\!(\eta, u\prod_{\zeta\in I} t_\zeta^{c_\zeta}, \sum_{\zeta\in I} c_\zeta b_\zeta)\!).$$

## 5.2 The proof

This proof relies on several ingredients: the relative Dehn method (Theorem 1.12), the local-to-global theorem for cosystolic expansion (Theorem 2.33), spectral expansion of the $B_3$-type Chevalley complexes (Theorem 4.30), and two new theorems:

**Theorem 5.4** (Bound for *fixed* $B_3$ links)**.** *$\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_5)$ and $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_5)$ are homologically 1-connected over $\mathbb{F}_2$ and $\mathbb{F}_3$.*

**Theorem 5.5** (Strong lifting theorem for $B_3$ links)**.** *Recall Definitions 1.4, 1.8 and 1.9. There exists $\delta, \ell, r_0\in\mathbb{N}$ such that for every odd prime (power) $p$* and every *$k\ge 1$:*

1. *Recall $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ and its subgroups $\mathcal{H}$. For every $\zeta\in\Phi_{B_3}^{\mathrm{sm},+}$, $t\in\mathbb{F}_{p^k}$, and $i\in[\mathrm{height}_{B_3}^{\mathrm{sm}}(\zeta)]$, there exists an "alias" word $\langle\!\langle\zeta, t, i\rangle\!\rangle$ of length at most $r_0$ over $\mathcal{H}$ evaluating to $(\!(\zeta, t, i)\!)$ in $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ such that the following holds. Consider the following three sets of relators over $\mathcal{H}$:*
   - *$\mathcal{R}^{\mathrm{St}}$, which is defined as the image of the set of pure-degree Steinberg relations in $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ under the "aliasing" mapping replacing each pure degree element $(\!(\zeta, t, i)\!)$ with the corresponding word $\langle\!\langle\zeta, t, i\rangle\!\rangle$.*

- $\mathcal{R}_\ell^{\mathrm{common}}$*, the set of in-subgroup relators in* $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ *of length at most* $\ell$ *(Definition 1.13).*
- $\mathcal{R}_\ell^{\mathrm{Lift}}$*, which is defined as the union, over all lifting homomorphisms* $\varphi$ *from* $U_{B_3}^{\mathrm{sm}}(\mathbb{F}_p)$ *to* $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ *(in the sense of Theorem 5.1), of the image of the set of all length-*$\ell$ *relators in* $U_{B_3}^{\mathrm{sm}}(\mathbb{F}_p)$ *(in the sense of Theorem 3.33).*

*Then every* $w \in \mathcal{R}^{\mathrm{St}}$ *has* $\mathrm{area}(w; \mathcal{R}_\ell^{\mathrm{common}} \cup \mathcal{R}_\ell^{\mathrm{Lift}}) \leq \delta$*.*

2. *The same holds for* $GU_{B_3}^{\mathrm{lg}}(\mathbb{F}_{p^k})$ *and* $\Phi_{B_3}^{\mathrm{lg},+}$*.*

See Theorems D.13 and D.14 for fully explicit (but quite long) versions of this theorem.

We prove Theorem 5.4 by explicitly analyzing the homology of the complexes $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_5)$ and $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_5)$ on a computer; we give details on how we did so in §6 below. Theorem 5.5 is essentially a group-theoretic result about the graded unipotent subgroups $GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_{p^k})$ and $GU_{B_3}^{\mathrm{lg}}(\mathbb{F}_{p^k})$. A proof of this theorem appeared in an earlier version of this work, but it was highly laborious and so we have omitted it in favor of referring to the Lean formalization which eventually appeared in [WBCS25].

Now, we put the pieces of the puzzle together and show how Theorems 1.12, 2.33, 4.30, 5.4 and 5.5 suffice to prove Theorem 1.19.

*Proof of Theorem 1.19.* Let $R_0 \coloneqq r_0 \cdot |\Phi|$, where $r_0$ is the fixed constant guaranteed by Proposition 4.20 (for $\Phi = B_3$, no dependence on $q$ or $m$). Let $\delta, \ell, r_0$ be the fixed constants guaranteed by Theorem 5.5. There by Theorem 2.33, there exists $\gamma > 0$ s.t. it suffices to check the following: For every $m \in \mathbb{N}$, $\mathfrak{B}_{5^k}^3(m)$ satisfies:

1. $\mathfrak{B}_{5^k}^3(m)$ is 0-connected.
2. All vertex-links in $\mathfrak{B}_q^3(m)$ are 0-connected.
3. All edge-links have 0-spectral expansion parameter at most $\gamma$.
4. Every vertex-link has coboundary expansion at least $\beta$ over $\Gamma$.

Items 1 and 2 follow from immediately Proposition 4.27. Item 3 follows from Theorem 4.30 assuming that we take $k$ to be sufficiently large so that $\frac{1}{\sqrt{5^k/2-2}} < \gamma$. The main challenge is therefore proving Item 4.

To prove Item 4, we apply the relative Dehn method (Theorem 1.12). All vertex-links in $\mathfrak{B}_q^3(m)$ are isomorphic to either $\mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ or $\mathfrak{GLB}_3^{\mathrm{lg}}(\mathbb{F}_q)$; we consider the case $\mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ WLOG.

Recall that $I = \{\beta, \psi, \omega\}$. Consider the group $G \coloneqq GU_{B_3}^{\mathrm{sm}}(\mathbb{F}_q) = GU_I(\mathbb{F}_q)$. Let $\mathcal{H}$ denote the corresponding $I$-indexed subgroup family $\mathcal{H} = (GU_{I \setminus \{\zeta\}})_{\zeta \in I}$ (cf. Definition 4.14). Hence $\mathfrak{X} \coloneqq \mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q) = \mathfrak{CC}(G; \mathcal{H})$ by definition.

We now apply Theorem 1.12 to $\mathfrak{X}$. By Propositions 4.16 and 4.20, every element of $G$ can be expressed as the evaluation of some word over $\mathcal{H}$ of length at most $R_0$. So, it suffices to check that every $w \in \mathcal{R}_{2R_0+1}$ satisfies $\mathrm{area}(w; \mathcal{R}_{\ell,t}()) \leq \delta'$, where $\mathcal{R}_{\ell,t}()$ is the set of $t$-fillable relators of length at most $\ell$ in $\mathfrak{X}$, and $t$ and $\delta'$ is a (large) constant of our choosing.

First, we apply Theorem 4.22 to conclude that all relators over $\mathcal{H}$ of any fixed length (in particular $2R_0 + 1$) can be derived in a fixed number $\delta_1'$ of steps from $\mathcal{R}^{\mathrm{St}}$. (Note that Theorem 4.22 uses elements which may not be in $\mathcal{H}$, but any derivation from Theorem 4.22 can be simulated using alias words which are over $\mathcal{H}$, at the cost of a multiplicative factor of $r_0'$ in the derivation length.) Next, we apply Theorem 5.5 to get that every relator in $\mathcal{R}^{\mathrm{St}}$ can be derived in a fixed number of steps from $\mathcal{R}_\ell^{\mathrm{common}} \cup \mathcal{R}_\ell^{\mathrm{Lift}}$. So, it suffices to check that $\mathcal{R}_\ell^{\mathrm{common}} \subseteq \mathcal{R}_{\ell,t}()$ and $\mathcal{R}_\ell^{\mathrm{Lift}} \subseteq \mathcal{R}_{\ell,t}()$. The former follows immediately from Fact 3.21. For the latter, we combine Theorems 3.33 and 5.4 and Proposition 3.22, and take $t$ to be the total number of triangles in the base complex $\mathfrak{GLB}_3^{\mathrm{sm}}(\mathbb{F}_q)$. ☐

# 6 Computational analysis for $B_3$

In this section, we discuss how we proved Theorem 5.4 via a computer analysis, and various other aspects of using computers to calculate the homology groups of Chevalley complexes (including how we checked that some Chevalley complexes are *not* homologically 1-connected over certain finite fields).

The computational analysis (proof of Theorem 5.4) is divided into a "pipeline" of two separate parts:

1. Calculating the triangle-edge incidence matrix of Chevalley link groups over finite fields $\mathbb{F}_q$.
2. Calculating the rank of an arbitrary sparse matrix over finite fields $\mathbb{F}_p$.

Together, these two parts let us calculate the first Betti numbers of the relevant complexes, and therefore check homological 1-connectedness over $\mathbb{F}_p$ (cf. Definition 2.19). (We emphasize that $p$ and $q$ may be different.) For the rank calculation, we used some prebuilt tools. We address these two parts in separate subsections.

## 6.1 Generating the incidence matrices

We wrote a C++ script (after prototyping in Sage) to generate the triangle-edge incidence matrices of the complexes $\mathfrak{L}\mathfrak{A}_3(\mathbb{F}_q)$, $\mathfrak{L}\mathfrak{B}_3^{\mathrm{sm}}(\mathbb{F}_q)$, and $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$ for arbitrary prime powers $q$.[23] (For the largest complex, $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$, this is only practical up to $q = 7$.) Here we describe how the script works for the $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$ case; the functionality for other complexes is similar. Recall that the base vectors for $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$ are denoted $\alpha$, $\beta$, and $\psi$.

1. We define a subroutine which, given a root vector $\zeta \in \Phi_{B_3}$ and coefficient $t \in \mathbb{F}_q$, computes the $7 \times 7$ matrix over $\mathbb{F}_q$ corresponding to the Chevalley group element $\{\{\zeta, t\}\}$.
2. We greedily enumerate an unordered set of all matrices in the unipotent group $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$. Initially, the set contains only the identity element $\mathbb{1}$, and then, while possible, we add new elements to the set by multiplying existing elements by the generators $\{\{\alpha, 1\}\}$, $\{\{\beta, 1\}\}$, and $\{\{\psi, 1\}\}$. (This set contains $q^9$ matrices, which is $\approx 1.95 \cdot 10^6$ when $q = 5$ and $\approx 40.6 \cdot 10^6$ when $q = 7$.) Each matrix (group element) corresponds to a *triangle* in the complex $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$.
3. We use a similar procedure to enumerate the *edges* in the complex:
   - The "red-green" edges correspond to cosets of the group generated by $\alpha$ elements in of $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$.
   - The "red-blue" edges correspond to cosets of the group generated by $\beta$ elements in $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$.
   - The "green-blue" edges correspond to cosets of the group generated by $\psi$ elements $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$.

   Each coset therefore contains $q$ matrices, and there are $q^8$ coset of each type.[24] We also construct dictionaries mapping elements of $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_q)$ to their three corresponding cosets, or equivalently, mapping triangles in $\mathfrak{L}\mathfrak{B}_3^{\mathrm{lg}}(\mathbb{F}_q)$ to their three corresponding edges.
4. We output the triangle-edge incidence matrix in the `sms` (sparse matrix storage) format in order to subsequently calculate their rank. We emphasize that this matrix only has $+1$ and $0$ entries, because we consistently orient all triangles as red-green-blue.

**Remark 6.1.** Recall that given some fixed ordering of the roots in a positive subset, every link group element can be expressed uniquely as a product of elements of each type in the link. If the link has $e$ roots, then we go from encoding group elements as $n \times n$ matrices over $\mathbb{F}_q$ ($n = 7$ in the case of $B_3$) to length-$t$ vectors over $\mathbb{F}_q$. This could be the basis for a more memory-efficient program to output the triangle-edge incidence matrix. This program also has the nice property that the set of triangles (a.k.a. group elements) is simply the product set $\mathbb{F}_q^e$. Further, e.g. the red-blue edges correspond to cosets of the subgroup $K_{\mathrm{RED}} \cap K_{\mathrm{BLUE}} \cong \mathbb{F}_q$ and so may be represented by length-$(e-1)$ vectors. Indeed, even calculating the red-blue edge incident to some triangle is just some polynomial mapping $\mathbb{F}_q^e \to \mathbb{F}_q^{e-1}$. However, we chose to keep the current matrix group approach as it is more "canonical" (does not require fixing orderings of the roots) and as computing the rank, not outputting the incidence matrix, is the bottleneck step. $\diamondsuit$

[23] This script will be uploaded on Github for the final version of this paper.

[24] We do not need to explicitly write down the *vertices* of the complex, but if we were to do so, the "red", "green", and "blue" vertices would correspond to cosets of the subgroups generated by $\alpha$ and $\beta$ elements, $\alpha$ and $\psi$ elements, and $\beta$ and $\psi$ elements, respectively. There would be $q^6$ red vertices, $q^7$ green vertices, and $q^5$ blue vertices.

| $q$ | $\lvert\mathfrak{X}(2)\rvert$ | $\lvert\mathfrak{X}(1)\rvert$ | $\lvert\mathfrak{X}(0)\rvert$ | $\mu(\mathfrak{X})$ | $\beta_1(\mathfrak{X};\mathbb{F}_2)$ | $\beta_1(\mathfrak{X};\mathbb{F}_3)$ | $\beta_1(\mathfrak{X};\mathbb{F}_5)$ | $\beta_1(\mathfrak{X};\mathbb{F}_7)$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 64 | 96 | 32 | 65 | 2 | 2 | 2 | 2 |
| 3 | 729 | 729 | 135 | 595 | 0 | 0 | 0 | 0 |
| 5 | 15,625 | 9,375 | 875 | 8,501 | 0 | 0 | 0 | 0 |
| 7 | 117,649 | 50,421 | 3,087 | 47,335 | 0 | 0 | 0 | 0 |

Table 1: First Betti numbers of the type-$A_3$ link complex $\mathfrak{X} \coloneqq \mathfrak{LA}_3(\mathbb{F}_q)$ over various small finite fields. The complex has $|\mathfrak{X}(2)| = q^6$ triangles, $|\mathfrak{X}(1)| = 3q^5$ edges, and $|\mathfrak{X}(0)| = q^4 + 2q^3$ vertices. Therefore, $\mu(\mathfrak{X}) = |\mathfrak{X}(1)| - |\mathfrak{X}(0)| + 1 = 3q^5 - (q^4 + 2q^3) + 1$.

| $q$ | $\lvert\mathfrak{X}(2)\rvert$ | $\lvert\mathfrak{X}(1)\rvert$ | $\lvert\mathfrak{X}(0)\rvert$ | $\mu(\mathfrak{X})$ | $\beta_1(\mathfrak{X};\mathbb{F}_2)$ | $\beta_1(\mathfrak{X};\mathbb{F}_3)$ | $\beta_1(\mathfrak{X};\mathbb{F}_5)$ | $\beta_1(\mathfrak{X};\mathbb{F}_7)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 2,187 | 2,187 | 351 | 1,837 | 12 | 12 | 12 | 12 |
| 5 | 78,125 | 46,875 | 3,875 | 43,001 | 0 | 0 | 2 | 0 |
| 7 | 823,543 | 352,947 | 19,551 | 333,397 | 0 | 0 | 0 | 2 |

Table 2: First Betti numbers of the small type-$B_3$ link complex $\mathfrak{X} \coloneqq \mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ over various small finite fields. The complex has $|\mathfrak{X}(2)| = q^7$ triangles, $|\mathfrak{X}(1)| = 3q^6$ edges, and $|\mathfrak{X}(0)| = q^5 + q^4 + q^3$ vertices. Therefore, $\mu(\mathfrak{X}) = |\mathfrak{X}(1)| - |\mathfrak{X}(0)| + 1 = 3q^6 - (q^5 + q^4 + q^3) + 1$. We also verified that for $q = 3$, $\beta_1(\mathfrak{X};\mathbb{F}_p) = 12$ for $p \in \{11, 13, 17, 19, 23, 27, 29\}$, and $q = 5$, $\beta_1(\mathfrak{X};\mathbb{F}_p) = 0$ for $p \in \{11, 13, 17\}$.

| $q$ | $\lvert\mathfrak{X}(2)\rvert$ | $\lvert\mathfrak{X}(1)\rvert$ | $\lvert\mathfrak{X}(0)\rvert$ | $\mu(\mathfrak{X})$ | $\beta_1(\mathfrak{X};\mathbb{F}_2)$ | $\beta_1(\mathfrak{X};\mathbb{F}_3)$ | $\beta_1(\mathfrak{X};\mathbb{F}_5)$ | $\beta_1(\mathfrak{X};\mathbb{F}_7)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 19,683 | 19,683 | 3,149 | 16,525 | 0 | 4 | 0 | 0 |
| 5 | 1,953,125 | 1,171,875 | 96,875 | 1,075,001 | 0 | 0 | 0 | ? |
| 7 | 40,353,607 | 17,294,403 | 957,999 | 16,336,405 | ? | ? | ? | ? |

Table 3: First Betti numbers of the large type-$B_3$ link complex $\mathfrak{X} \coloneqq \mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_q)$ over various small finite fields. The complex has $|\mathfrak{X}(2)| = q^9$ triangles, $|\mathfrak{X}(1)| = 3q^8$ edges, and $|\mathfrak{X}(0)| = q^7 + q^6 + q^5$ vertices. Therefore, $\mu(\mathfrak{X}) = |\mathfrak{X}(1)| - |\mathfrak{X}(0)| + 1 = 3q^8 - (q^7 + q^6 + q^5) + 1$.

### 6.2 Computing the rank of the matrices over $\mathbb{F}_p$

Let $q$ be a small prime (typically 2, 3, 5, or 7). As mentioned in the previous subsection, in the complex $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_q)$, the "large link" of $B_3$ over $\mathbb{F}_q$, there are $q^9$ triangles, $3q^8$ edges ($q^8$ for each possible color-pair), and $q^7 + q^6 + q^5$ vertices. The triangle-edge incidence matrix of this complex is a $q^9 \times 3q^8$ matrix with three 1's per row,[25] and we managed to calculate its rank over various small finite fields $\mathbb{F}_p$ for small values of $q$. See Table 3 for the results, and Tables 1 and 2 for similar results for the complexes $\mathfrak{LA}_3(\mathbb{F}_q)$ and $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_q)$. To make the presentation more readable, we enumerate the *first Betti numbers* $\beta_1(\cdot;\mathbb{F}_p)$. Recall that a complex is homologically 1-connected over $\mathbb{F}_p$ iff its first Betti number over $\mathbb{F}_p$ is 0, and that the first Betti number of $\mathfrak{X}$ over $\Gamma$ equals $\mu(\mathfrak{X}) - \mathrm{rank}(\delta^1;\Gamma)$.

(Note that the vanishing of homology is already guaranteed for $\mathfrak{LA}_3(\mathbb{F}_q)$ by the absolute Dehn method and algebraic calculations in [KO21], but we calculated the ranks for this complex as a sanity check.)

**Details of the rank calculation.** For highly sparse matrices such as these, using Gaussian elimination to calculate the rank can be tricky: In an $m \times n$ matrix with $O(1)$ nonzero entries per row, there are only $O(m)$ total nonzero entries, but an unlucky choice of pivots for Gaussian elimination can cause *fill-in*, making the resulting matrix dense, with $\Omega(m^2)$ nonzero entries. When $m \sim 10^6$ and especially when $m \sim 10^7$, with our computing resources we can only afford $\Theta(m)$ memory (megabytes), not $\Theta(m^2)$ memory (terabytes).

[25] As mentioned earlier, we can pick triangle and edge orientations such that no $-1$ entries are needed.

However, we did manage to perform the rank calculation, even for $\mathfrak{GLB}_3^{\mathrm{lg}}(\mathbb{F}_q)$, in the $q = 5$ case. (This is why Theorem 5.4 and therefore our main application theorem Theorem 1.19 are only stated for this case.) Recall, the $q = 5$ triangle-edge incidence matrix is $1{,}953{,}125 \times 1{,}171{,}875$, so this rank computation was quite expensive. We used the software package `linbox` [Lin18], which implements sparse Gaussian elimination over finite fields. This computation already took roughly 10 hours to run on our personal computers; $q = 7$ would be impractical, given the huge blowup in time and memory usage. We also received independent confirmation of the rank from an optimized program for calculating ranks of sparse matrices over $\mathbb{F}_2$ due to Ryan Bai and Richard Peng (personal communication).[26]

**Interpretations of tables.** We use that $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_5)$ and $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_5)$ are *both* homologically 1-connected over $\mathbb{F}_2$ and $\mathbb{F}_3$. In contrast, $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_5)$ is *not* homologically 1-connected over $\mathbb{F}_5$ (though $\mathfrak{LB}_3^{\mathrm{lg}}(\mathbb{F}_5)$ is), meaning it does not have coboundary expansion over $\mathbb{F}_5$. Similarly, $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_7)$ is not homologically 1-connected over $\mathbb{F}_5$, and $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_3)$ is not homologically 1-connected over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, or $\mathbb{F}_7$. In particular, these imply that $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_3)$, $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_5)$, and $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_7)$ are not simply connected.

**Conjecture 6.2.** *We conjecture that for every odd prime power $q$, $\mathfrak{LB}_3^{\mathrm{sm}}(\mathbb{F}_q)$ is never homologically 1-connected over $\mathbb{F}_q$, and therefore that it is not simply connected.*

# References


[ABN23] Anurag Anshu, Nikolas P. Breuckmann, and Chinmay Nirkhe. “NLTS Hamiltonians from Good Quantum Codes”. In: *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*. STOC 2023 (Orlando, FL, USA). Association for Computing Machinery, June 2, 2023, pp. 1090–1096. DOI: 10.1145/3564246.3585114.

[Adi57] S. I. Adian. “Unsolvability of some algorithmic problems in the theory of groups”. In: *Trudy Moskovskogo Matematicheskogo Obshchestva* 6 (1957), pp. 231–298. DOI: 10.2307/2269670.

[AH93] H. Abels and S. Holz. “Higher Generation by Subgroups”. In: *Journal of Algebra* 160.2 (Oct. 1993), pp. 310–341. DOI: 10.1006/jabr.1993.1190.

[Alo86] Noga Alon. “Eigenvalues and Expanders”. In: *Combinatorica* 6.2 (June 1986), pp. 83–96. DOI: 10.1007/BF02579166.

[ALOV24] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. “Log-Concave Polynomials II: High-dimensional Walks and an FPRAS for Counting Bases of a Matroid”. In: *Annals of Mathematics* 199.1 (Jan. 1, 2024). Conference version in STOC 2019. DOI: 10.4007/annals.2024.199.1.4.

[AOV21] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. “Log-Concave Polynomials, I: Entropy and a Deterministic Approximation Algorithm for Counting Bases of Matroids”. In: *Duke Mathematical Journal* 170.16 (Nov. 1, 2021). Conference version in FOCS 2018. DOI: 10.1215/00127094-2020-0091.

[BD01] Daniel K Biss and Samit Dasgupta. “A Presentation for the Unipotent Group over Rings with Identity”. In: *Journal of Algebra* 237.2 (Mar. 2001), pp. 691–707. DOI: 10.1006/jabr.2000.8604.

[BLM24] Mitali Bafna, Noam Lifshitz, and Dor Minzer. “Constant Degree Direct Product Testers with Small Soundness”. In: *65th Annual Symposium on Foundations of Computer Science*. FOCS 2024 (Chicago, IL, USA, Oct. 27–30, 2024). IEEE Computer Society, July 17, 2024, pp. 862–869. DOI: 10.1109/FOCS61266.2024.00059.

[BM24] Mitali Bafna and Dor Minzer. “Characterizing Direct Product Testing via Coboundary Expansion”. In: *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*. STOC 2024 (Vancouver, BC, Canada, June 24–28, 2024). Association for Computing Machinery, Feb. 1, 2024, pp. 1978–1989. DOI: 10.1145/3618260.3649714.

[BMVY25] Mitali Bafna, Dor Minzer, Nikhil Vyas, and Zhiwei Yun. “Quasi-Linear Size PCPs with Small Soundness from HDX”. In: *Proceedings of the 57th Annual ACM Symposium on Theory of Computing*. STOC 2025 (Prague, Czechia, June 23–27, 2025). New York, NY, USA: Association for Computing Machinery, June 15, 2025, pp. 45–53. DOI: 10.1145/3717823.3718197.


[26] This program also uses sparse Gaussian elimination. The idea behind their script is to perform Gaussian elimination on the matrix in sparse representation until it becomes sufficiently dense, and then to switch to a dense matrix representation which uses a bitarray. Even with a simple pivoting heuristic — eliminate the rows with the smallest number of nonzeros — this seemed to run about 10x faster than `linbox`.

[Bri02] Martin R. Bridson. “The Geometry of the Word Problem”. In: *Invitations to Geometry and Topology*. Ed. by Martin R. Bridson and Simon M. Salamon. Oxford Graduate Texts in Mathematics 7. Oxford, Oct. 3, 2002, pp. 29–91. DOI: 10.1093/oso/9780198507727.003.0002.

[BS92] László Babai and Mario Szegedy. “Local Expansion of Symmetrical Graphs”. In: *Combinatorics, Probability and Computing* 1.1 (Mar. 1992), pp. 1–11. DOI: 10.1017/S0963548300000031.

[BŚ97] W. Ballmann and J. Świątkowski. “On L2-cohomology and Property (T) for Automorphism Groups of Polyhedral Cell Complexes”. In: *Geometric and Functional Analysis* 7.4 (Aug. 1997), pp. 615–645. DOI: 10.1007/s000390050022.

[Bun52] Svend Bundgaard. “On a Kind of Homotopy in Regular Numbered Complexes”. In: *Meddelanden från Lunds Universitets Matematiska Seminarium* Tome Suppl. (1952), pp. 35–46.

[Car89] Roger W. Carter. *Simple Groups of Lie Type*. London: Wiley, 1989. 335 pp. ISBN: 978-0-471-50683-6.

[Chu96] Fan Chung. *Spectral Graph Theory*. Vol. 92. CBMS Regional Conference Series in Mathematics. Providence, Rhode Island: American Mathematical Society, Dec. 3, 1996. DOI: 10.1090/cbms/092.

[CL25] Michael Chapman and Alexander Lubotzky. “Stability of Homomorphisms, Coverings and Cocycles II: Examples, Applications and Open Problems”. In: *Advances in Mathematics* 463 (Mar. 2025), p. 110117. DOI: 10.1016/j.aim.2025.110117.

[DD24a] Yotam Dikstein and Irit Dinur. “Coboundary and Cosystolic Expansion Without Dependence on Dimension or Degree”. In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. RANDOM 2024 (London, UK, Aug. 28–30, 2024). Ed. by Amit Kumar and Noga Ron-Zewi. Vol. 317. LIPIcs. 2024, 62:1–62:24. DOI: 10.4230/LIPICS.APPROX/RANDOM.2024.62.

[DD24b] Yotam Dikstein and Irit Dinur. “Swap Cosystolic Expansion”. In: *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*. STOC 2024 (Vancouver, BC, Canada, June 24–28, 2024). Association for Computing Machinery, June 10, 2024, pp. 1956–1966. DOI: 10.1145/3618260.3649780.

[DDL24] Yotam Dikstein, Irit Dinur, and Alexander Lubotzky. “Low Acceptance Agreement Tests via Bounded-Degree Symplectic HDXs”. In: *65th Annual Symposium on Foundations of Computer Science*. FOCS 2024 (Chicago, IL, USA, Oct. 27–30, 2024). IEEE Computer Society, Oct. 27, 2024, pp. 826–861. DOI: 10.1109/FOCS61266.2024.00058.

[DEL+22] Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. “Locally Testable Codes with Constant Rate, Distance, and Locality”. In: *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2022 (Rome, Italy, June 20–24, 2022). Association for Computing Machinery, June 9, 2022, pp. 357–374. DOI: 10.1145/3519935.3520024.

[DFHT21] Irit Dinur, Yuval Filmus, Prahladh Harsha, and Madhur Tulsiani. “Explicit SoS Lower Bounds from High-Dimensional Expanders”. In: *12th Innovations in Theoretical Computer Science Conference*. ITCS 2021 (virtual, Jan. 6–8, 2021). 2021. DOI: 10.4230/LIPIcs.ITCS.2021.38.

[DK17] Irit Dinur and Tali Kaufman. “High Dimensional Expanders Imply Agreement Expanders”. In: *IEEE 58th Annual Symposium on Foundations of Computer Science*. FOCS 2017. Oct. 2017, pp. 974–985. DOI: 10.1109/FOCS.2017.94.

[DKW18] Dominic Dotterrer, Tali Kaufman, and Uli Wagner. “On Expansion and Topological Overlap”. In: *Geometriae Dedicata* 195.1 (Aug. 2018). Conference version in SoCG 2016, pp. 307–317. DOI: 10.1007/s10711-017-0291-4.

[EK16] Shai Evra and Tali Kaufman. “Bounded Degree Cosystolic Expanders of Every Dimension”. In: *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing*. STOC 2016 (Cambridge, MA, USA, June 19–21, 2016). Association for Computing Machinery, June 19, 2016, pp. 36–48. DOI: 10.1145/2897518.2897543.

[EKZ24] Shai Evra, Tali Kaufman, and Gilles Zémor. “Decodable Quantum LDPC Codes beyond the $\sqrt{n}$ Distance Barrier Using High-Dimensional Expanders”. In: *SIAM Journal on Computing* 53.6 (Dec. 2024). Conference version in FOCS 2020, FOCS20-276-FOCS20–316. DOI: 10.1137/20M1383689.

[Eve78] Martha Evens. “A Program for the Todd–Coxeter Coset Enumeration Algorithm”. 1978.

[Gar79] Peter Freedman Garst. “Cohen-Macaulay Complexes and Group Actions”. PhD thesis. University of Wisconsin-Madison, May 17, 1979. 144 pp.

[Gro10] Mikhail Gromov. “Singularities, Expanders and Topology of Maps. Part 2: From Combinatorics to Topology Via Algebraic Isoperimetry”. In: *Geometric and Functional Analysis* 20.2 (Aug. 2010), pp. 416–526. DOI: 10.1007/s00039-010-0073-8.

[GV25] Laura Grave de Peralta and Inga Valentiner-Branth. “High-Dimensional Expanders from Kac–Moody–Steinberg Groups”. In: *European Journal of Combinatorics* 126 (May 2025), p. 104131. DOI: 10.1016/j.ejc.2025.104131.

[Hal03] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*. Vol. 222. Graduate Texts in Mathematics. New York, NY: Springer New York, 2003. DOI: 10.1007/978-0-387-21554-9.

[HL22] Max Hopkins and Ting-Chun Lin. “Explicit Lower Bounds Against $\Omega(n)$-Rounds of Sum-of-Squares”. In: *63rd Annual Symposium on Foundations of Computer Science*. FOCS 2022 (Denver, CO, USA, Oct. 31–Nov. 3, 2022). IEEE Computer Society, 2022. DOI: 10.1109/FOCS54457.2022.00069.

[HLM+25a] Jun-Ting Hsieh, Ting-Chun Lin, Sidhanth Mohanty, Ryan O’Donnell, and Rachel Yun Zhang. “Explicit Two-Sided Vertex Expanders beyond the Spectral Barrier”. In: *Proceedings of the 57th Annual ACM Symposium on Theory of Computing*. STOC 2025 (Prague, Czechia). Association for Computing Machinery, June 15, 2025, pp. 833–842. DOI: 10.1145/3717823.3718241.

[HLM+25b] Jun-Ting Hsieh, Alexander Lubotzky, Sidhanth Mohanty, Assaf Reiner, and Rachel Yun Zhang. “Explicit Lossless Vertex Expanders”. In: *66th Annual Symposium on Foundations of Computer Science*. FOCS 2025 (Sydney, Australia, Dec. 14–17, 2025). IEEE Computer Society, Apr. 21, 2025.

[HS24] Prahladh Harsha and Ramprasad Saptharishi. “On the Elementary Construction of High-Dimensional Expanders by Kaufman and Oppenheim”. In: *Theory of Computing* 20.1 (2024), pp. 1–22. DOI: 10.4086/toc.2024.v020a005.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Vol. 9. Graduate Texts in Mathematics. New York, NY: Springer New York, 1972. DOI: 10.1007/978-1-4612-6398-2.

[Kir78] John Frederick Kirchmeyer. “Minimal Presentations for the Unipotent Subgroups of Certain Chevalley Groups”. PhD thesis. Evanston, IL, USA: Northwestern University, Aug. 1978. 90 pp.

[KKL16] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. “Isoperimetric Inequalities for Ramanujan Complexes and Topological Expanders”. In: *Geometric and Functional Analysis* 26.1 (Feb. 1, 2016). Conference version in FOCS 2014, pp. 250–287. DOI: 10.1007/s00039-016-0362-y.

[KM19] Dmitry N. Kozlov and Roy Meshulam. “Quantitative Aspects of Acyclicity”. In: *Research in the Mathematical Sciences* 6.4 (Dec. 2019), p. 33. ISSN: 2522-0144, 2197-9847. DOI: 10.1007/s40687-019-0195-z. URL: http://link.springer.com/10.1007/s40687-019-0195-z (visited on 09/04/2025).

[KO18] Tali Kaufman and Izhar Oppenheim. “Construction of New Local Spectral High Dimensional Expanders”. In: *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2018 (Los Angeles, CA, USA, June 25–29, 2018). Association for Computing Machinery, June 20, 2018, pp. 773–786. DOI: 10.1145/3188745.3188782.

[KO21] Tali Kaufman and Izhar Oppenheim. “Coboundary and Cosystolic Expansion from Strong Symmetry”. In: *48th International Colloquium on Automata, Languages, and Programming*. ICALP 2021 (Glasgow, Scotland / virtual, July 12–16, 2021). Ed. by Nikhil Bansal, Emanuela Merelli, and James Worrell. Vol. 198. LIPIcs. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021, 84:1–84:16. DOI: 10.4230/LIPICS.ICALP.2021.84.

[KOW25] Tali Kaufman, Izhar Oppenheim, and Shmuel Weinberger. “Coboundary Expansion of Coset Complexes”. In: *57th Annual ACM Symposium on Theory of Computing*. STOC 2025 (Prague, Czechia, June 23–27, 2025). Association for Computing Machinery, June 15, 2025, pp. 1722–1731. DOI: 10.1145/3717823.3718136.

[Lan50] Folke Lannér. “On Complexes with Transitive Groups of Automorphisms”. In: *Meddelanden från Lunds Universitets Matematiska Seminarium* 11 (1950).

[Lin18] The LinBox Group. *LinBox*. Version 1.6.3. 2018. URL: http://github.com/linbox-team/linbox.

[LMM16] Alexander Lubotzky, Roy Meshulam, and Shahar Mozes. “Expansion of Building-like Complexes”. In: *Groups, Geometry, and Dynamics* 10.1 (Feb. 9, 2016), pp. 155–175. DOI: 10.4171/ggd/346.

[OP22] Ryan O’Donnell and Kevin Pratt. “High-Dimensional Expanders from Chevalley Groups”. In: *LIPIcs, Volume 234, CCC 2022*. 37th Computational Complexity Conference (Philadelphia, PA, USA, July 20–23, 2022). Ed. by Shachar Lovett. Vol. 234. LIPIcs. CCC 2022: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 18:1–18:26. DOI: 10.4230/LIPICS.CCC.2022.18.

[Opp18] Izhar Oppenheim. “Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps”. In: *Discrete & Computational Geometry* 59.2 (Mar. 2018), pp. 293–330. DOI: 10.1007/s00454-017-9948-x.

[PK22] Pavel Panteleev and Gleb Kalachev. "Asymptotically Good Quantum and Locally Testable Classical LDPC Codes". In: *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2022. New York, NY, USA: Association for Computing Machinery, June 9, 2022, pp. 375–388. DOI: 10.1145/3519935.3520017.

[Pra23] Kevin Pratt. "Hypergraph Rank and Expansion". PhD thesis. Carnegie Mellon University, Sept. 27, 2023. 140 pp.

[Rab58] Michael O. Rabin. "Recursive Unsolvability of Group Theoretic Problems". In: *The Annals of Mathematics* 67.1 (Jan. 1958), p. 172. ISSN: 0003486X. DOI: 10.2307/1969933. JSTOR: 1969933. URL: https://www.jstor.org/stable/1969933?origin=crossref (visited on 09/05/2025).

[Ril17] Tim Riley. "Office Hour Eight: Dehn Functions". In: *Office Hours with a Geometric Group Theorist*. Ed. by Matt Clay and Dan Margalit. Princeton University Press, July 11, 2017, pp. 146–175. DOI: 10.2307/j.ctt1vwmg8g.

[Ste16] Robert Steinberg. *Lectures on Chevalley Groups*. Vol. 66. University Lecture Series. Providence, Rhode Island: American Mathematical Society, Dec. 20, 2016. DOI: 10.1090/ulect/066.

[WBCS25] Eric Wang, Arohee Bhoja, Cayden Codel, and Noah G. Singer. "Algebra Is Half the Battle: Verifying Presentations for Graded Unipotent Chevalley Groups". In: *16th International Conference on Interactive Theorem Proving*. ITP 2025 (Reykjavik, Iceland, Sept. 27–Oct. 3, 2025). LIPIcs. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025.

# A Why $A_3$'s link is simply connected and $B_3$'s might not be

In this appendix, we discuss why in the $A_3$-type setting, [KO21] were able to prove that $\mathfrak{LA}_3(\mathbb{F}_p)$ and $\mathfrak{GLA}_3(\mathbb{F}_p)$ are simply connected for all odd prime powers $p$, while in the $B_3$ setting, we conjecture that such a theorem is not true (Conjecture 6.2).

For $\mathfrak{LA}_3(\mathbb{F}_p)$, the [KO21] result is equivalent (via the absolute Dehn method) to stating that all Steinberg relations in $U_{A_3}(\mathbb{F}_{\mathbb{F}_p})$ can be derived from in-subgroup relations in $U_{A_3}(\mathbb{F}_{\mathbb{F}_p})$, in a number of steps which is independent of $p$. This is due essentially to Biss and Dasgupta [BD01].[27] The key step of the [BD01] proof (essentially the only difficult one) is to derive the relation "$\alpha+\beta$ and $\beta+\gamma$ elements commute". (This makes sense as a first step because it is the *only* relation in the $A_3$ case which does not name the "missing" root $\alpha+\beta+\gamma$.) We give our own shorter (and perhaps conceptually clearer) proof of this relation, and then discuss why we do *not* know how to prove analogous relations in $B_3$.

## A.1 An alternative proof of Biss–Dasgupta

In this section we give an alternative proof of the key Biss–Dasgupta result [BD01, Sec. 4], which states that over the ring $\mathbb{Z}/p\mathbb{Z}$ with $p$ odd, the fact that $(\alpha+\beta)$-type and $(\beta+\gamma)$-type elements commute can be derived only using "in-subgroup relations". In fact, our proof works within the unipotent group $U_{A_3}(R)$ for any ring $R$ in which $\frac{1}{2}$ exists (i.e., $1+1$ is a unit).

Our proof, as in the original proof in [BD01, §4] proof, starts by deriving the following useful "rewriting" relation in $U_{A_3}(R)$:

**Relation A.1.** *For any ring $R$, whenever the elements $\frac{uv}{t+v}$ and $\frac{tu}{t+v}$ exist, one can derive*

$$\{\{\alpha,t\}\}\{\{\beta,u\}\}\{\{\alpha,v\}\} = \{\{\beta,\tfrac{uv}{t+v}\}\}\{\{\alpha,t+v\}\}\{\{\beta,\tfrac{tu}{t+v}\}\}$$

*using only in-subgroup relations in $U_{A_3}(R)$, and the number of steps in the proof does not depend on $R$. The same holds replacing $\alpha$'s with $\beta$'s and $\beta$'s with $\gamma$'s.*

*Proof.* First, note that $u = \frac{uv}{t+v} + \frac{tu}{t+v}$. Thus, we can write, by linearity of $\beta$ elements:

$$\begin{aligned}&\{\{\alpha,t\}\}\{\{\beta,u\}\}\{\{\alpha,v\}\}\\ &= \{\{\alpha,t\}\}\{\{\beta,\tfrac{uv}{t+v}\}\}\{\{\beta,\tfrac{tu}{t+v}\}\}\{\{\alpha,v\}\}\end{aligned}$$

[27] The original [BD01] proof did not emphasize the quantitative bound on the length of the derivation, and in particular, the fact that the length of the derivation does *not* depend on the size $p$ of the ring.

And by the definition of the commutator:

$$= \{\{\beta, \tfrac{uv}{t+v}\}\}\{\{\alpha, t\}\} \left[\{\{\alpha, t\}\}^{-1}, \{\{\beta, \tfrac{uv}{t+v}\}\}^{-1}\right] \left[\{\{\beta, \tfrac{tu}{t+v}\}\}, \{\{\alpha, v\}\}\right]^{-1} \{\{\alpha, v\}\}\{\{\beta, \tfrac{tu}{t+v}\}\}.$$

Using the fact that inverting $\alpha$, $\beta$, and $\alpha+\beta$ elements is the same as negating the respective entries, and that the commutator of $\alpha$ and $\beta$ elements is an $\alpha+\beta$ element whose product is the element of the entries, these two commutators cancel (they are $\alpha+\beta$ elements with entries $t\frac{uv}{t+v}$ and $-t\frac{uv}{t+v}$, respectively. Linearity of $\alpha$ elements gives the desired relation. ☐

Now, we can prove:

**Relation A.2.** *In any ring $R$ containing $\frac{1}{2}$, one can derive that*

$$\forall t, u \in R, [\{\{\alpha+\beta, t\}\}, \{\{\beta+\gamma, u\}\}] = \mathbb{1}$$

*using only in-subgroup relations in $U_{A_3}(R)$, and the number of steps in the proof does not depend on $R$.*

*Proof.* For this proof, we need some subgroup identities. We use four identities which expand elements as commutators (using, again, that inverting a $\zeta$ element negates its entry for $\zeta \in \{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma\}$). These identities require that $\frac{1}{2}$ exist in $R$:

$$\{\{\alpha+\beta, t\}\} = \{\{\beta, -1\}\}\{\{\alpha, t\}\}\{\{\beta, 1\}\}\{\{\alpha, -t\}\}, \tag{A.3}$$

$$\{\{\beta+\gamma, u\}\} = \{\{\gamma, -2u\}\}\{\{\beta, \tfrac{1}{2}\}\}\{\{\gamma, 2u\}\}\{\{\beta, -\tfrac{1}{2}\}\}, \tag{A.4}$$

$$\{\{\alpha+\beta, -t\}\} = \{\{\beta, \tfrac{1}{2}\}\}\{\{\alpha, 2t\}\}\{\{\beta, -\tfrac{1}{2}\}\}\{\{\alpha, -2t\}\}, \tag{A.5}$$

$$\{\{\beta+\gamma, -u\}\} = \{\{\gamma, -u\}\}\{\{\beta, -1\}\}\{\{\gamma, u\}\}\{\{\beta, 1\}\}. \tag{A.6}$$

We also require four identities for swaps following from Relation A.1; again, these require $\frac{1}{2}$ to exist in $R$:

$$\{\{\alpha, -t\}\}\{\{\beta, \tfrac{1}{2}\}\}\{\{\alpha, 2t\}\} = \{\{\beta, 1\}\}\{\{\alpha, t\}\}\{\{\beta, -\tfrac{1}{2}\}\}, \tag{A.7}$$

$$\{\{\gamma, 2u\}\}\{\{\beta, -\tfrac{1}{2}\}\}\{\{\gamma, -u\}\} = \{\{\beta, \tfrac{1}{2}\}\}\{\{\gamma, u\}\}\{\{\beta, -1\}\}, \tag{A.8}$$

$$\{\{\beta, 1\}\}\{\{\gamma, -2u\}\}\{\{\beta, 1\}\} = \{\{\gamma, -u\}\}\{\{\beta, 2\}\}\{\{\gamma, -u\}\}, \tag{A.9}$$

$$\{\{\beta, -1\}\}\{\{\alpha, -2t\}\}\{\{\beta, -1\}\} = \{\{\alpha, -t\}\}\{\{\beta, -2\}\}\{\{\alpha, -t\}\}. \tag{A.10}$$

The basic plan is now to write down $[\{\{\alpha+\beta, t\}\}, \{\{\beta+\gamma, u\}\}]$ and expand using Equations (A.3) to (A.6). We proceed as follows:

$$\underline{[\{\{\alpha+\beta, t\}\}, \{\{\beta+\gamma, u\}\}]}$$

Expand the commutators:

$$= \{\{\alpha+\beta, t\}\}\underline{\{\{\beta+\gamma, u\}\}\{\{\alpha+\beta, -t\}\}}\{\{\beta+\gamma, -u\}\}$$

Expand with Equations (A.4) and (A.5):

$$= \{\{\alpha+\beta, t\}\}\{\{\gamma, -2u\}\}\{\{\beta, \tfrac{1}{2}\}\}\{\{\gamma, 2u\}\}\underline{\{\{\beta, -\tfrac{1}{2}\}\}\{\{\beta, \tfrac{1}{2}\}\}}\{\{\alpha, 2t\}\}\{\{\beta, -\tfrac{1}{2}\}\}\{\{\alpha, -2t\}\}\{\{\beta+\gamma, -u\}\}$$

Cancel the $\beta$ elements:

$$= \underline{\{\{\alpha+\beta, t\}\}}\{\{\gamma, -2u\}\}\{\{\beta, \tfrac{1}{2}\}\}\{\{\gamma, 2u\}\}\{\{\alpha, 2t\}\}\{\{\beta, -\tfrac{1}{2}\}\}\{\{\alpha, -2t\}\}\underline{\{\{\beta+\gamma, -u\}\}}$$

Expand with Equations (A.3) and (A.6):

$$\begin{aligned} = \{\{\beta, -1\}\}\{\{\alpha, t\}\}\{\{\beta, 1\}\}&\underline{\{\{\alpha, -t\}\}\{\{\gamma, -2u\}\}}\{\{\beta, \tfrac{1}{2}\}\}\underline{\{\{\gamma, 2u\}\}} \\ &\underline{\cdot\{\{\alpha, 2t\}\}}\{\{\beta, -\tfrac{1}{2}\}\}\underline{\{\{\alpha, -2t\}\}\{\{\gamma, -u\}\}}\{\{\beta, -1\}\}\{\{\gamma, u\}\}\{\{\beta, 1\}\} \end{aligned}$$

Commute $\alpha$ and $\gamma$ elements where possible:

$$= \{\{\beta,-1\}\}\{\{\alpha,t\}\}\{\{\beta,1\}\}\{\{\gamma,-2u\}\}\underline{\{\{\alpha,-t\}\}\{\{\beta,\tfrac{1}{2}\}\}\{\{\alpha,2t\}\}}$$
$$\underline{\cdot\{\{\gamma,2u\}\}\{\{\beta,-\tfrac{1}{2}\}\}\{\{\gamma,-u\}\}}\{\{\alpha,-2t\}\}\{\{\beta,-1\}\}\{\{\gamma,u\}\}\{\{\beta,1\}\}$$

Swap $\alpha\cdot\beta\cdot\alpha$ products for $\beta\cdot\alpha\cdot\beta$ products and similarly for $\gamma$ and $\beta$ (Equations (A.7) and (A.8)):

$$= \{\{\beta,-1\}\}\{\{\alpha,t\}\}\{\{\beta,1\}\}\{\{\gamma,-2u\}\}\{\{\beta,1\}\}\{\{\alpha,t\}\}\underline{\{\{\beta,-\tfrac{1}{2}\}\}}$$
$$\underline{\cdot\{\{\beta,\tfrac{1}{2}\}\}}\{\{\gamma,u\}\}\{\{\beta,-1\}\}\{\{\alpha,-2t\}\}\{\{\beta,-1\}\}\{\{\gamma,u\}\}\{\{\beta,1\}\}$$

Cancel adjacent $\beta$ elements:

$$= \{\{\beta,-1\}\}\{\{\alpha,t\}\}\underline{\{\{\beta,1\}\}\{\{\gamma,-2u\}\}\{\{\beta,1\}\}}\{\{\alpha,t\}\}\{\{\gamma,u\}\}\underline{\{\{\beta,-1\}\}\{\{\alpha,-2t\}\}\{\{\beta,-1\}\}}\{\{\gamma,u\}\}\{\{\beta,1\}\}$$

Do another swap (Equations (A.9) and (A.10)):

$$= \{\{\beta,-1\}\}\{\{\alpha,t\}\}\{\{\gamma,-u\}\}\{\{\beta,2\}\}\underline{\{\{\gamma,-u\}\}\{\{\alpha,t\}\}\{\{\gamma,u\}\}\{\{\alpha,-t\}\}}\{\{\beta,-2\}\}\{\{\alpha,-t\}\}\{\{\gamma,u\}\}\{\{\beta,1\}\}$$

Commute $\alpha$ and $\gamma$ elements and therefore cancel them:

$$= \{\{\beta,-1\}\}\{\{\alpha,t\}\}\{\{\gamma,-u\}\}\underline{\{\{\beta,2\}\}\{\{\beta,-2\}\}}\{\{\alpha,-t\}\}\{\{\gamma,u\}\}\{\{\beta,1\}\}$$

Cancel $\beta$ elements:

$$= \{\{\beta,-1\}\}\underline{\{\{\alpha,t\}\}\{\{\gamma,-u\}\}\{\{\alpha,-t\}\}\{\{\gamma,u\}\}}\{\{\beta,1\}\}$$

Again commute and cancel $\alpha$ and $\gamma$ elements:

$$= \{\{\beta,-1\}\}\{\{\beta,1\}\}$$

Finally cancel $\beta$ elements:

$$= \mathbb{1},$$

as desired. □

Incidentally, the fact that Relation A.2 holds implies that in any link complex $\mathfrak{L}\mathfrak{A}_3(\mathbb{F}_q)$, there must be an $\Gamma$-filling of the loop corresponding to the critical relation $[\{\{\alpha+\beta,t\}\},\{\{\beta+\gamma,u\}\}]=\mathbb{1}$. We found a simple (human-verifiable) proof of this fact, illustrated by the diagrams in Figure 6. The $[\{\{\alpha+\beta,t\}\},\{\{\beta+\gamma,u\}\}]=\mathbb{1}$ relation corresponds to a loop of length 4, alternating between RED and BLUE subgroups; this is the orange outer loop in the figures. In the coset complex, some (but not all) RED - BLUE - RED - BLUE "squares" have the property that they are $\Gamma$-filled by 4 triangles, all incident on the same GREEN vertex. Although the orange outer loop does *not* have this property, we find five other squares that *are* fillable in this way, and such that the $\Gamma$-boundary of these squares is the orangle loop to be filled. The arrangement of these squares is depicted in the left figure, and the full filling by triangles is depicted in the right figure. Overall, this gives an $\Gamma$-filling of the critical relation $[\{\{\alpha+\beta,t\}\},\{\{\beta+\gamma,u\}\}]=\mathbb{1}$ using 20 triangles.

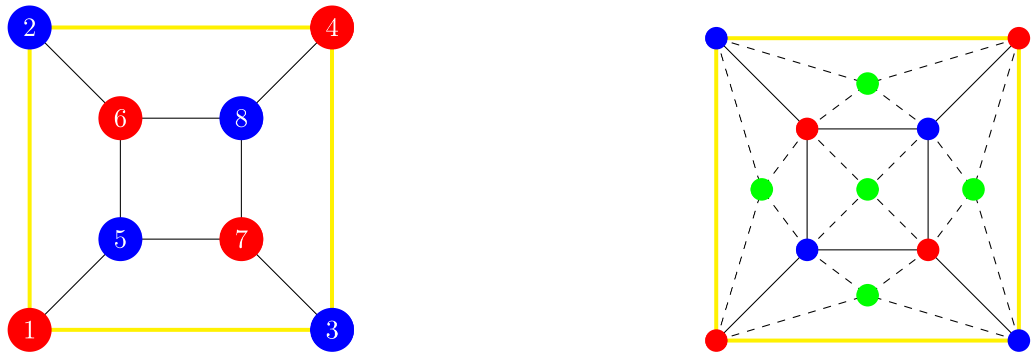



Figure 6: Depiction of the $\mathbb{F}_2$-filling of relation $[\{\{\alpha+\beta,1\}\},\{\{\beta+\gamma,1\}\}] = \mathbb{1}$ in $\mathfrak{LA}_3(\mathbb{F}_q)$. In the left-hand figure, the vertices are labeled $1,\dots,8$. These correspond to the following cosets: $H_\alpha$, $H_\gamma$, $\{\{\alpha+\beta,1\}\}H_\alpha$, $\{\{\beta+\gamma,1\}\}H_\gamma$, $\{\{\beta,1\}\}\{\{\gamma,2\}\}H_\gamma$, $\{\{\beta,1\}\}\{\{\alpha,1\}\}H_\alpha$, $\{\{\alpha+\beta,1\}\}\{\{\beta,2\}\}\{\{\gamma,1\}\}H_\gamma$, and $\{\{\beta+\gamma,1\}\}\{\{\beta,\frac{1}{2}\}\}\{\{\gamma,2\}\}H_\gamma$, respectively.

### A.2 $A_3$ vs. $B_3$

Let us now reflect on some key features of $U_{A_3}(\mathbb{F}_q)$ that let us write this proof. Firstly, $\alpha+\beta$ and $\beta+\gamma$ were expressible as $\alpha$-$\beta$ and $\beta$-$\gamma$ commutators, respectively. By using factors of 2 and $\frac{1}{2}$ in two of the expansions (Equations (A.4) and (A.5)) but not the others (Equations (A.3) and (A.6)), we were able to introduce some asymmetries allowing us to eventually swap $\eta\zeta\eta$ type products for $\zeta\eta\zeta$ type products. We also got mileage out of the complete symmetry between the pairs $\alpha,\beta$ and $\beta,\gamma$.

We tried extensively to find a similar derivation for the group $U^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$: A proof, using only in-subgroup relations, that $\beta+\psi$ and $\psi+\omega$ commute. However, the situation there is not as nice: $\beta+\psi$ is not the only root in the positive span of $\beta$ and $\psi$: $\beta+2\psi$ is as well! Thus, $\beta+\psi$ elements are not commutators of $\beta$ and $\psi$ elements. They can be expressed as $\psi\beta\psi\beta\psi$ products, but the entries on the first and last $\psi$ elements in this expansion are the same, so we have less asymmetry to play with.

We also attempted to replicate the $A_3$ success depicted in Figure 6; that is, we used a computer to look for reasonably small, "nicely structured" fillings of the cycle corresponding to the commutation relation for $\beta+\psi$ and $\psi+\omega$ roots in $\mathfrak{LB}^{\mathrm{sm}}_3(\mathbb{F}_5)$. Unfortunately, we concluded that there is no comparably small and symmetric filling.

## B The homology cones method for (commutative) rings

In this appendix, we prove Theorem 2.36, restated as follows:

**Theorem 2.36** (Cones method, generalizing [KO21, Thm. 3.8])**.** *Let $\Gamma$ be a ring and let $\mathfrak{X}$ be a 2-dimensional simplicial complex. Assume that $\mathfrak{X}$ is strongly symmetric (and therefore $\pi_2$ is the uniform distribution on $\mathfrak{X}(2)$). If $R_0$ is the diameter of $\mathfrak{X}$ and $\mathfrak{X}$ is $(2R_0+1, R_1)$-homologically taut over $\Gamma$, then $\mathfrak{X}$ has 1-coboundary expansion at least $1/R_1$ over $\Gamma$.*

Our proof generally follows that of [KO21], but it works for general coefficient rings $\Gamma$ (and also is shorter, as it is specialized to the 1-dimensional case).

In order to prove Theorem 2.36, we need a bit of setup. Firstly, we use some simple facts about sampling in strongly symmetric complexes:

**Fact B.1.** *Let $\mathfrak{X}$ be a strongly symmetric 2-dimensional complex. For every fixed triangle $t \in \mathfrak{X}(2)$ and $\boldsymbol{\varphi} \sim \mathrm{Aut}(\mathfrak{X})$ uniformly, $\boldsymbol{\varphi}t$ is distributed uniformly on $\mathfrak{X}(2)$.*

*Proof.* By the orbit-stabilizer theorem, for any group $G$ acting on any set $S$, for fixed $x \in S$ and uniformly random $\boldsymbol{g} \sim G$, $\boldsymbol{g}x$ is uniformly random on the orbit $Gx$ of $x$. In particular, if the action of $G$ on $S$ is transitive, $Gx = S$ and so $\boldsymbol{g}x$ is uniform on $S$. We apply this to the case where $G = \mathrm{Aut}(\mathfrak{X})$ and $x = t$. $\square$

**Fact B.2.** *Let $\mathfrak{X}$ be a strongly symmetric 2-dimensional complex. For every fixed $\varphi \in \mathrm{Aut}(\mathfrak{X})$ and $\boldsymbol{e} \sim \vec{\pi}_1$, $\varphi^{-1}\boldsymbol{e}$ is also distributed according to $\vec{\pi}_1$.*

*Proof.* Consider the distribution of $\varphi^{-1}\boldsymbol{e}$ when $\boldsymbol{e} \sim \pi_1$. By definition of $\pi_1$, this distribution is equivalent to the following process: Sample $\boldsymbol{t} \sim \mathfrak{X}(2)$ uniformly, sample $\boldsymbol{e} \subset \boldsymbol{t}$ uniformly, and output $\varphi^{-1}\boldsymbol{e}$. Equivalently, one can sample $\boldsymbol{t} \sim \mathfrak{X}(2)$ uniformly, sample $\boldsymbol{e} \subset \varphi\boldsymbol{t}$ uniformly, and output $\boldsymbol{e}$. Moreover, $\varphi\boldsymbol{t}$ is distributed uniformly on $\mathfrak{X}(2)$ (since $\varphi$ is an automorphism of $\mathfrak{X}$ and therefore a bijection on $\mathfrak{X}(2)$). So, we might as well sample $\boldsymbol{t} \sim \mathfrak{X}(2)$ uniformly, sample $\boldsymbol{e} \subset \boldsymbol{t}$ uniformly, and output $\boldsymbol{e}$. But this is just sampling from $\pi_1$. $\square$

For two 1-chains $f, g \in C^1(\mathfrak{X};\Gamma)$, we define an inner product

$$\langle f, g\rangle \coloneqq \sum_{e \in \mathfrak{X}(1)} f(e) \cdot g(e).$$

(There is a slight abuse of notation here: The sum is over undirected edges $\{u, v\}$, but $f$ and $g$ are defined on directed edges $(u, v)$. However, this does not cause a problem, because by antisymmetry, $f(v, u)g(v, u) = f(u, v)g(u, v)$.) We similarly define, for two 2-chains $S, T \in C^2(\mathfrak{X};\Gamma)$, an inner product

$$\langle S, T\rangle \coloneqq \sum_{t \in \mathfrak{X}(2)} f(t) \cdot g(t).$$

We have the standard adjointness identity: For every $f \in C^1(\mathfrak{X};\Gamma)$ and $T \in C^2(\mathfrak{X};\Gamma)$,

$$\langle f, \partial_2 T\rangle = \langle \delta^1 f, T\rangle. \tag{B.3}$$

When $W = (v_0 \to \cdots \to v_\ell)$ is a walk and $f$ a 1-cochain, we define

$$\Sigma_W f \coloneqq \langle [W]_\Gamma, f\rangle = \sum_{i=1}^{\ell} f(v_{i-1}, v_i)$$

(we ignore terms where $v_{i-1} = v_i$). Hence $\Sigma_W f + \Sigma_{W'} f = \Sigma_{W \circ W'} f$ (assuming $W'$ begins where $W$ ends) and $-\Sigma_W f = \Pi_{W^{-1}} f$.

The idea behind the cones method is now as follows. If there were some perfectly explanatory 0-cochain (vertex-assignment) $h$ for $f$, we would have $f(u, v) = h(v) - h(u)$ for every edge $(u, v) \in \vec{\mathfrak{X}}(1)$. Hence for any walk $W$ from $u$ to $v$, we would also have $\Sigma_W f = h(v) - h(u)$ by telescoping. We can therefore make a guess $h'$ for an explanatory assignment by setting $h'(u)$ arbitrarily and $h(v) \coloneqq h'(u) + \Sigma_W v$ given a walk $W$ from $u$ to $v$. (If the walks $W$ are picked via breadth-first search, this process can be viewed as "propagating" the guessed value $h'(u)$ along the walk $W$ by setting $h'(v_1) = f(v_0, v_1) + h'(v_0)$, $h'(v_2) = f(v_1, v_2) + h'(v_1)$, and so on.) If a perfectly explanatory assignment exists (and $\mathfrak{X}$ is connected, so that walks exist), this procedure will produce an $h'$ which perfectly explanatory. Indeed, if the paths are short, this could potentially still produce a mostly explanatory assignment even if no perfectly explanatory assignment exists. We formalize this intuition in the following.

To analyze these propagated assignments, we will be interested in quantities $\Sigma_W f$ where $L$ is a loop and $f$ a 1-cochain; this can be viewed as a sum of $f$ "around" the loop $L$. (In analogy to differential geometry, for a loop $L$, this quantity can be viewed as a measure of the *circulation* or *holonomy* of $f$ around the loop $L$.) The following key claim is a sufficient condition for a 1-cochain to "vanish" around translations of a loop:

**Lemma B.4.** *Let $L$ be a loop . Then there exists a set of triangles $T \subseteq \mathfrak{X}(2)$ of size $|T| \le \triangle^{\Gamma}_{\mathrm{H}}(L)$ such that for every $\varphi \in \mathrm{Aut}(\mathfrak{X})$ and 1-cochain $f \in C^1(\mathfrak{X};\Gamma)$, if $\varphi T \cap \mathrm{supp}(\delta^1 f) = \emptyset$, then $\Sigma_{\varphi L} f = 0$.*

*Proof.* Let $S \in C^2(\mathfrak{X};\Gamma)$ be a minimum-size $\Gamma$-triangulation of $L$; that is, $\partial_2 S = [L]_\Gamma$ and $|\mathrm{supp}(S)| = \triangle^{\Gamma}_{\mathrm{H}}(L)$. We set $T \coloneqq \mathrm{supp}(S)$. For every $\varphi \in G$, $\partial_2(\varphi S) = \varphi(\partial_2 S) = \varphi[L]_\Gamma = [\varphi L]_\Gamma$. Hence by adjointness, $\Sigma_{\varphi L} f = \langle \varphi S, \delta^1 f\rangle$. This indeed vanishes whenever $\mathrm{supp}(\varphi S) = \varphi T$ is disjoint from $\mathrm{supp}(\delta^1(f))$, as desired. $\square$

Consequently, strong symmetry gives:

**Corollary B.5.** *Let $L$ be a loop and $f \in C^1(\mathfrak{X};\Gamma)$ a 1-cochain. Then for $\boldsymbol{\varphi} \sim \mathrm{Aut}(\mathfrak{X})$ uniformly:*

$$\Pr_{\boldsymbol{\varphi}}[\Sigma_{\boldsymbol{\varphi}L} f \neq 0] \leq \triangle^{\Gamma}_{\mathrm{H}}(L) \cdot \mathrm{dist}(\delta^1 f, 0).$$

*Proof.* Using the prior lemma and Markov's inequality,

$$\Pr_{\boldsymbol{\varphi}}[\Sigma_{\boldsymbol{\varphi}L} f \neq 0] \leq \Pr_{\boldsymbol{\varphi}}[\boldsymbol{\varphi}T \cap \mathrm{supp}(\delta^1 f) \neq \emptyset] \leq \sum_{t \in \mathrm{supp}(\delta^1 f)} \sum_{t' \in T} \Pr_{\boldsymbol{\varphi}}[\boldsymbol{\varphi}t = t'] = \left|\mathrm{supp}(\delta^1 f)\right| \cdot |T| \cdot \frac{1}{|\mathfrak{X}(2)|}$$

where we used that $\Pr_{\boldsymbol{\varphi}}[\boldsymbol{\varphi}t = t'] = \frac{1}{|\mathfrak{X}(2)|}$ from Fact B.1. □

Using this proposition, we now prove:

*Proof of Theorem 2.36.* Pick a vertex $u$ and, for every vertex $v \in \mathfrak{X}(0)$, a walk $P_v$ from $u$ to $v$. (By assumption on the diameter of $\mathfrak{X}$, $|P_v| \leq R_0$.) For $\varphi \in \mathrm{Aut}(\mathfrak{X})$, define a "rotated" walk $P_v^{\varphi} \coloneqq \varphi P_{\varphi^{-1}v}$, also from $u$ to $v$.

Using these walks, we define a collection of 0-cochains (i.e., vertex-labelings) $h_\varphi \in C^0(\mathfrak{X};\Gamma)$:

$$h_\varphi(v) \coloneqq \Sigma_{P_v^{\varphi}} f.$$

We show that in expectation over $\boldsymbol{\varphi} \sim \mathrm{Aut}(\mathfrak{X})$ uniformly, $h_{\boldsymbol{\varphi}}(v)$ has low error in explaining $f$.

To do so, we give a condition equivalent to $(\delta^0 h)(e) = f(e)$. For every edge $(v,w) \in \vec{\mathfrak{X}}(1)$, define the loop $L_{(v,w)} \coloneqq P_v \circ (v,w) \circ P_w^{-1}$. We therefore have $L^{\varphi}_{(v,w)} \coloneqq \varphi L_{\varphi^{-1}(v,w)} = P_v^{\varphi} \circ (v,w) \circ (P_w^{\varphi})^{-1}$. Observe that:

$$(\delta^0 h)(v,w) = f(v,w) \iff h_\varphi(w) - h_\varphi(v) = f(v,w) \iff \Sigma_{P_w^{\varphi}} f - \Sigma_{P_v^{\varphi}} f = \Sigma_{(v \to w)} f \iff \Sigma_{L^{\varphi}_{(v,w)}} f = 0.$$

Hence (sampling $\boldsymbol{\varphi} \sim \mathrm{Aut}(\mathfrak{X})$ uniformly and $\boldsymbol{e} \sim \vec{\pi}_1$), we have:

$$\mathop{\mathbb{E}}_{\boldsymbol{\varphi}}[\mathrm{dist}(\delta^0 h_{\boldsymbol{\varphi}}, f)] = \Pr_{\boldsymbol{\varphi},\boldsymbol{e}}[(\delta^0 h_{\boldsymbol{\varphi}})(\boldsymbol{e}) \neq f(\boldsymbol{e})] = \Pr_{\boldsymbol{\varphi},\boldsymbol{e}}[\Sigma_{L^{\boldsymbol{\varphi}}_{\boldsymbol{e}}} f \neq 0] = \Pr_{\boldsymbol{e},\boldsymbol{\varphi}}[\Sigma_{\boldsymbol{\varphi}L_{\boldsymbol{e}}} f \neq 0] \leq R_1 \cdot \mathrm{dist}(\delta^1 f, 0)$$

where the third equality uses Fact B.2 (and that $L^{\boldsymbol{\varphi}}_{\boldsymbol{e}} = \varphi L_{\boldsymbol{\varphi}^{-1}\boldsymbol{e}}$), and the inequality uses Corollary B.5, that $|L_e| \leq 2R_0 + 1$, and the assumed tautness of $\mathfrak{X}$. □

# C The absolute Dehn method

In this appendix, we reprove the *absolute Dehn method* of Kaufman and Oppenheim [KO21] using the *nonabelian cones method* of Dikstein and Dinur [DD24b]. See Theorem C.23 below for a formal theorem statement, which achieves slightly improved quantitative parameters.

## C.1 Defining homotopy area

We define a notion of "homotopy area" based on equivalence relations on loops. This is essentially taken from [DD24b, §4], with the (very) minor technical difference that we allow repeated vertices in our walks, and do not confine ourselves to walks with a single basepoint:

**Definition C.1.** Let $\mathfrak{X}$ be a simplicial complex. We define a "trivial" equivalence relation $\overset{0}{\sim}$ on the set of loops in $\mathfrak{X}$ as the smallest equivalence relation with the following property: If $W_1$ is a walk from $u$ to $v$, $B$ a backtracking loop at $v$, and $W_2$ a walk from $v$ to $u$, then $W_1 \circ W_2 \overset{0}{\sim} W_1 \circ B \circ W_2$. (Note that if $L_1 \overset{0}{\sim} L_2$ then $L_1$ and $L_2$ have the same basepoint.) ◊

**Fact C.2.** *Suppose $L$ and $L'$ are two loops at $v$ and $L \overset{0}{\sim} L'$. Then:*

1. *$L^{-1} \overset{0}{\sim} (L')^{-1}$.*
2. *For every $\varphi \in \mathrm{Aut}(\mathfrak{X})$, $\varphi L \overset{0}{\sim} \varphi L'$.*
3. *If $W_1$ is a walk from $u$ to $v$ and $W_2$ a walk from $v$ to $u$, then $W_1 \circ L \circ W_2 \overset{0}{\sim} W_1 \circ L' \circ W_2$.*

*Proof.* In all three proofs, we induct on the length of the equivalence chain connecting $L$ and $L'$; the base case is that $L = L'$ in which case all three equalities hold trivially. For the inductive step, we show the desired equivalences for $L = W_1 \circ W_2$ and $L' = W_1 \circ B \circ W_2$ (where $B$ is some backtracking loop). We verify:

$$(L')^{-1} = (W_1 \circ B \circ W_2)^{-1} = W_2^{-1} \circ B^{-1} \circ W_1^{-1} \overset{0}{\sim} W_2^{-1} \circ W_1^{-1} = (W_1 \circ W_2)^{-1} = (L)^{-1},$$
$$\varphi L' = \varphi(W_1 \circ B \circ W_2) = \varphi W_1 \circ \varphi B \circ \varphi W_2 \overset{0}{\sim} \varphi W_1 \circ \varphi W_2 = \varphi(W_1 \circ W_2) = \varphi L,$$
$$W_1' \circ L' \circ W_2' = W_1' \circ W_1 \circ B \circ W_2 \circ W_2' \overset{0}{\sim} W_1' \circ W_1 \circ W_2 \circ W_2' = W_1' \circ L \circ W_2'. \qquad \square$$

**Definition C.3.** Let $L$ and $L'$ be two loops in $\mathfrak{X}$. We write $L \overset{1}{\sim} L'$ if there exist some $\{v, w, x\} \in \mathfrak{X}(2)$ and walks $W_1$ from $u$ to $v$ and $W_2$ from $w$ to $u$ such that $L \overset{0}{\sim} W_1 \circ (v \to x \to w) \circ W_2$ and $L' \overset{0}{\sim} W_1 \circ (v \to w) \circ W_2$ or vice versa. $\Diamond$

In algebraic topology, equivalence classes of loops under the transitive closure of $\overset{1}{\sim}$ are called *homotopy classes*.

**Fact C.4.** *Suppose $L$ and $L'$ are two loops at $v$ and $L \overset{1}{\sim} L'$. Then:*

1. $L^{-1} \overset{1}{\sim} (L')^{-1}$.
2. *For every* $\varphi \in \mathrm{Aut}(\mathfrak{X})$, $\varphi L \overset{1}{\sim} \varphi L'$.
3. *If $W_1$ is a walk from $u$ to $v$ and $W_2$ a walk from $v$ to $u$, then* $W_1 \circ L \circ W_2 \overset{1}{\sim} W_1 \circ L' \circ W_2$.

*Proof.* Suppose WLOG that $L \overset{0}{\sim} W_1 \circ (v \to x \to w) \to W_2$ and $L' \overset{0}{\sim} W_1 \circ (v \to w) \circ W_2$. We verify:

$$(L')^{-1} \overset{0}{\sim} W_2^{-1} \circ (w \to v) \circ W_1^{-1} \overset{1}{\sim} W_2^{-1} \circ (w \to x \to v) \circ W_1^{-1} \overset{0}{\sim} L^{-1},$$
$$\varphi L' \overset{0}{\sim} \varphi W_1 \circ (\varphi v \to \varphi w) \circ \varphi W_2 \overset{1}{\sim} \varphi W_1 \circ (\varphi v \to \varphi x \to \varphi w) \circ \varphi W_2 \overset{0}{\sim} \varphi L,$$
$$W_1' \circ L' \circ W_2' \overset{0}{\sim} W_1' \circ W_1 \circ (v \to w) \circ W_2 \circ W_2' \overset{1}{\sim} W_1' \circ W_1 \circ (v \to x \to w) \circ W_2 \circ W_2' \overset{0}{\sim} W_1' \circ L \circ W_2'. \qquad \square$$

**Definition C.5.** Let $\mathfrak{X}$ be a simplicial complex and $L$ a loop in $\mathfrak{X}$. We define $\triangle_\pi(L)$, the *homotopy area* of $L$, as the minimum $T$ such that there exist loops $L_0, \dots, L_T$ such that $L_0$ is an empty (length-0) loop, $L_T = L$, and $L_0 \overset{1}{\sim} \cdots \overset{1}{\sim} L_T$. $\Diamond$

**Proposition C.6.** *The area function $\triangle_\pi$ defined in Definition C.5 satisfies the axioms in Definition 3.2.*

*Proof.* The backtracking walk property follows from the fact that $W \circ W' \overset{0}{\sim} W \circ (v \to v) \circ W^{-1} \overset{0}{\sim} (v)$, an empty loop. Reversal and translation symmetries and subadditivity follow from Definition C.5 and iterated application of Fact C.4. The unit area property follows from the fact that by definition, $(u \to v \to w \to u) \overset{1}{\sim} (u \to v \to u \to u)$, which is a backtracking walk and therefore has area 0.

Finally, for cyclic symmetry, if $W_1$ is a walk from $u$ to $v$ and $W_2$ a walk from $v$ to $u$, then $W_2 \circ W_1 \overset{0}{\sim} W_2 \circ W_1 \circ W_2 \circ W_2^{-1}$. Hence by subadditivity and the backtracking property, $\triangle(W_2 \circ W_1) \leq \triangle(W_1 \circ W_2) + \triangle(W_2 \circ W_2^{-1}) = \triangle(W_1 \circ W_2)$. $\square$

We also introduce:

**Definition C.7.** Let $\mathfrak{X}$ be a $d$-dimensional simplicial complex ($d \geq 2$). We say that $\mathfrak{X}$ is $(R_0, R_1)$-*homotopically taut* if for every proper loop $L$ of length at most $R_0$, $\triangle_\pi(L) \leq R_1$. $\Diamond$

## C.2 Non-abelian coboundary expansion

In this appendix, we let $\Gamma$ denote an arbitrary multiplicative group. $\Gamma$ need not admit a ring structure; indeed, it need not even be abelian.

We define coboundary

**Definition C.8.** For $\mathfrak{X}$ a $d$-dimensional complex and $\Gamma$ any group. A 0-*cochain* $f$ in $\mathfrak{X}$ is an arbitrary function $f : \mathfrak{X}(0) \to \Gamma$. A 1-*cochain* $f$ in $\mathfrak{X}$ is a function $f : \vec{\mathfrak{X}}(1) \to \Gamma$ satisfying the antisymmetry property that for every $(u, v) \in \mathfrak{X}(1)$, $f(u, v) = f(v, u)^{-1}$. A 2-*cochain* $f$ in $\mathfrak{X}$ is a function $f : \vec{\mathfrak{X}}(2) \to \Gamma$ satisfying the antisymmetry property that for every $(u, v, w) \in \mathfrak{X}(2)$, $f(u, v, w) = f(v, w, u) = f(w, u, v) = f(v, u, w)^{-1} = f(u, w, v)^{-1} = f(w, v, u)^{-1}$. We let $C^j(\mathfrak{X}; \Gamma)$ denote the set of all $j$-cochains for $j \in \{0, 1, 2\}$; this set admits a (possibly nonabelian) group structure via (pointwise) multiplication. $\diamondsuit$

Again, as a group $C^j(\mathfrak{X}; \Gamma)$ is isomorphic to the $|\mathfrak{X}(j)|$-fold direct sum of $\Gamma$ (though the isomorphism depends on choice of orientations).

**Definition C.9.** For $\mathfrak{X}$ a $d$-dimensional complex, the 0-*coboundary operator* $\delta^0 : C^0(\mathfrak{X}; \Gamma) \to C^1(\mathfrak{X}; \Gamma)$ is defined on 0-cochains $f$ by

$$(\delta^0 f)(u, v) \coloneqq f(u)^{-1} f(v)$$

for every $(u, v) \in \vec{\mathfrak{X}}(1)$. The 1-*coboundary operator* $\delta^1 : C^1(\mathfrak{X}; \Gamma) \to C^2(\mathfrak{X}; \Gamma)$ is a homomorphism defined on 1-cochains $f$ by

$$(\delta^1 f)(u, v, w) \coloneqq f(u, v) f(v, w) f(w, u)$$

for every $(u, v, w) \in \vec{\mathfrak{X}}(2)$. $\diamondsuit$

**Fact C.10.** *For every $f \in C^0(\mathfrak{X}; \Gamma)$, $\delta^1(\delta^0 f) = \mathbb{1}$.*

*Proof.* Calculate

$$(\delta^1(\delta^0 f))(u, v, w) = ((\delta^0 f)(u, v))((\delta^0 f)(v, w))((\delta^0 f)(w, u)) = (f(u)^{-1} f(v))(f(v)^{-1} f(w))(f(w)^{-1} f(u)) = \mathbb{1}.$$

$\square$

**Definition C.11.** A 1-*cocycle* is a 1-chain $f \in C^1(\mathfrak{X}; \Gamma)$ such that $\delta^1 f = \mathbb{1}$; that is, for every oriented triangle $(u, v, w) \in \vec{\mathfrak{X}}(2)$,

$$f(u, v) f(v, w) f(w, u) = \mathbb{1}.$$

The 1-cocycles form a subgroup $Z^1(\mathfrak{X}; \Gamma) \coloneqq \ker \delta^1 \subseteq C^1(\mathfrak{X}; \Gamma)$. Similarly, the group of 1-coboundaries is $B^1(\mathfrak{X}; \Gamma) \coloneqq \operatorname{im} \delta^0 \subseteq Z^1(\mathfrak{X}; \Gamma) \subseteq C^1(\mathfrak{X}; \Gamma)$. $\diamondsuit$

**Definition C.12.** The 1-*st cohomology group* (over $\Gamma$) is the quotient group $H^1(\mathfrak{X}; \Gamma) \coloneqq Z^1(\mathfrak{X}; \Gamma)/B^1(\mathfrak{X}; \Gamma)$. The complex $\mathfrak{X}$ is *homologically* 1-*connected over* $\Gamma$ (or "has trivial 1-homology over $\Gamma$") if $B^1(\mathfrak{X}; \Gamma) = Z^1(\mathfrak{X}; \Gamma)$ (equiv. $H^1(\mathfrak{X}; \Gamma) = 0$). $\diamondsuit$

**Definition C.13.** For $(\mathfrak{X}, \pi)$ a weighted $d$-dimensional complex and $\Gamma$ a group, the *distance* between two $j$-chains $f$ and $f'$ is defined to be

$$\operatorname{dist}(f, f') \coloneqq \Pr_{\boldsymbol{\sigma} \sim \pi_j} [f(\boldsymbol{\sigma}) \neq f'(\boldsymbol{\sigma})].$$

The *weight* of a $j$-chain is $\operatorname{wt}(f) \coloneqq \operatorname{dist}(f, \mathbb{1})$. $\diamondsuit$

Again, we need not specify an orientation for $\boldsymbol{\sigma}$.

**Definition C.14.** Suppose that whenever $f$ is a 1-cochain over $\Gamma$ with $\operatorname{dist}(f, Z^1(\mathfrak{X}; \Gamma)) > v$, it holds that $\operatorname{dist}(\delta^1 f, 0) > \epsilon \cdot v$. Then we say that $\mathfrak{X}$ has 1-*cocycle expansion (at least)* $\epsilon$ *over* $\Gamma$, and we write $h^1(\mathfrak{X}; \Gamma)$ for the least possible such $\epsilon$. $\diamondsuit$

**Definition C.15.** We define the 1-*cosystole* to be

$$s^1(\mathfrak{X}; \Gamma) \coloneqq \min\{\operatorname{wt}(f) : f \in Z^1(\mathfrak{X}; \Gamma) \setminus B^1(\mathfrak{X}; \Gamma)\}, \tag{C.16}$$

with the convention $s^1(\mathfrak{X}; \Gamma) = 1$ if the 1-th cohomology vanishes over $\Gamma$. $\diamondsuit$

**Definition C.17.** Suppose the 1-cocycle expansion over $\Gamma$ satisfies $h^1(\mathfrak{X}; \Gamma) \geq \epsilon$. If, moreover, $s^1(\mathfrak{X}; \Gamma) \geq \mu$, we say that $\mathfrak{X}$ has 1-*cosystolic expansion (at least)* $(\epsilon, \mu)$ *over* $\Gamma$. If the 1-th cohomology in fact vanishes, $\mathfrak{X}$ is said to have 1-*coboundary expansion (at least)* $\epsilon$. $\diamondsuit$

## C.3 Non-abelian cones

We now state the following theorem:

**Theorem C.18** (essentially [DD24b, Lemma 1.6])**.** *Let $\mathfrak{X}$ be a 2-dimensional complex. Assume that $\mathfrak{X}$ is strongly symmetric (and therefore $\pi_2$ is the uniform distribution on $\mathfrak{X}(2)$). If $R_0$ is the diameter of $\mathfrak{X}$ and $\mathfrak{X}$ is $(2R_0+1, R_1)$-homologically taut over $\Gamma$, then $\mathfrak{X}$ has 1-coboundary expansion at least $1/R_1$ over $\Gamma$.*

This theorem is essentially proven in [DD24b, §4], but we reprove it here because of minor technical differences: We allow repeated vertices in our walks, and do not require them to maintain a fixed basepoint. (Also, the proof is very similar to our preceding proof of the homology cones method.)

For a walk $W = (v_0 \to \cdots \to v_\ell)$ and a 1-cochain $f \in C^1(\mathfrak{X};\Gamma)$, we define

$$\Pi_W f = \prod_{i=1}^{\ell} f(v_{i-1}, v_i)$$

(where we simply discard the terms where $v_{i-1} = v_i$). Hence $(\Pi_W f)\cdot(\Pi_{W'} f) = \Pi_{W\circ W'} f$ (assuming $W'$ begins where $W$ ends) and $(\Pi_W f)^{-1} = \Pi_{W^{-1}} f$.

**Fact C.19.** *Let $L, L'$ be loops and $f \in C^1(\mathfrak{X};\Gamma)$. If $L \overset{0}{\sim} L'$, then $\Pi_f L = \Pi_f L'$.*

*Proof.* Similarly to the proof of Fact C.2, we can induct on the length of the equivalence chain connecting $L$ and $L'$. If $L = L'$ then the equality is trivial. Otherwise, inductively, we need to check the equality for the case $L = W_1 \circ W_2$ and $L' = W_1 \circ B \circ W_2$ (where $B$ is some backtracking loop). Indeed,

$$\Pi_L f = (\Pi_{W_1} f)\cdot(\Pi_{W_2} f), \text{ while } \Pi_{L'} f = (\Pi_{W_1} f)\cdot(\Pi_B f)\cdot(\Pi_{W_2} f).$$

Hence, it is sufficient (and necessary) to check that $\Pi_B f = \mathbb{1}$ for every backtracking loop $B$. Indeed, either $B = W \circ W^{-1}$ or $B = W \circ (v \to v) \circ W^{-1}$; in either case, $\Pi_B f = (\Pi_W f)\cdot(\Pi_{W^{-1}} f) = (\Pi_W f)\cdot(\Pi_W f)^{-1} = \mathbb{1}$. □

**Fact C.20.** *Let $L, L'$ be loops and suppose that $L \overset{1}{\sim} L'$. Then there exists a triangle $t \in \mathfrak{X}(2)$ such that for every $\varphi \in \mathrm{Aut}(\mathfrak{X})$ and 1-cochain $f \in C^1(\mathfrak{X};\Gamma)$, if $(\delta^1 f)(\varphi t) = \mathbb{1}$, then $\Pi_{\varphi L} f = \Pi_{\varphi L'} f$.*

*Proof.* Suppose that $L \overset{0}{\sim} W_1 \circ (v \to x \to w) \circ W_2$ and $L' \overset{0}{\sim} W_1 \circ (v \to w) \circ W_2$ for some triangle $\{v,w,x\} \in \mathfrak{X}(2)$. Hence $\varphi L \overset{0}{\sim} \varphi W_1 \circ (\varphi v \to \varphi x \to \varphi w) \circ \varphi W_2$ and $\varphi L' \overset{0}{\sim} \varphi W_1 \circ (\varphi v \to \varphi w) \circ W_2$. We define $t \coloneqq \{v,x,w\}$. Similarly to the previous proof,

$$\Pi_{\varphi L} f = (\Pi_{\varphi W_1} f)\cdot(\Pi_{(\varphi v \to \varphi x \to \varphi w)} f)\cdot(\Pi_{\varphi W_2} f), \text{ while } \Pi_{L'} f = (\Pi_{\varphi W_1} f)\cdot(\Pi_{(\varphi v \to \varphi w)} f)\cdot(\Pi_{\varphi W_2} f).$$

Therefore, we need to check that if $(\delta^1 f)(\varphi t) = \mathbb{1}$, then $\Pi_{(\varphi v \to \varphi x \to \varphi w)} f = \Pi_{(\varphi v \to \varphi w)} f$. Indeed, the LHS is $f(\varphi v, \varphi x) f(\varphi x, \varphi w)$ and the RHS is $f(\varphi v, \varphi w)$; these are equal iff $(\delta^1 f)(\varphi t) = f(\varphi v, \varphi x) f(\varphi x, \varphi w) f(\varphi w, \varphi v) = \mathbb{1}$. □

**Lemma C.21.** *Let $L$ be a loop. Then there exists a set of triangles $T \subseteq \mathfrak{X}(2)$ of size $|T| \le \triangle_\pi(\Gamma)$ such that for every $\varphi \in \mathrm{Aut}(\mathfrak{X})$ and 1-cochain $f \in C^1(\mathfrak{X};\Gamma)$, if $\varphi T \cap \mathrm{supp}(\delta^1 f) = \emptyset$, then $\Pi_{\varphi L} f = \mathbb{1}$.*

*Proof.* Apply the previous fact iteratively together with the definition of homotopy area (Definition C.5). □

**Lemma C.22.** *Let $L$ be a loop and $f$ a 1-cochain. Then for $\boldsymbol{\varphi} \sim \mathrm{Aut}(\mathfrak{X})$ uniformly and every group $\Gamma$:*

$$\Pr_{\boldsymbol{\varphi}}[\Pi_{\boldsymbol{\varphi} L} f \neq \mathbb{1}] \le \triangle_\pi(L)\cdot \mathrm{dist}(\delta^1 f, \mathbb{1}).$$

*Proof.* Using the previous fact, exactly the same as the proof of Corollary B.5. □

Using this proposition, Theorem C.18 now follows exactly as Theorem 2.36 did (but with addition replaced with multiplication, negation by inversion, and $\Sigma$'s by $\Pi$'s.)

### C.4 Statement of the method

**Theorem C.23** (Absolute Dehn method)**.** *Consider the $d$-dimensional simplicial coset complex $\mathfrak{CC}(G;\mathcal{H})$. Suppose that there exist $R_0, \delta \in \mathbb{N}$ such that:*

1. *every element in $G$ can be written as the evaluation of a length-($\leq R_0$) word over $\mathcal{H}$.*

2. *every $w \in \mathcal{R}_{2R_0+1}$ satisfies $\operatorname{area}(w; \mathcal{R}_3^{\text{common}}) \leq \delta$.*

*Then $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$ and is $(2R_0+1, O(\delta))$-homotopically taut, and therefore has 1-coboundary expansion at least $\Omega(\frac{1}{\delta})$ over $\Gamma$ for every $\Gamma$ (including nonabelian groups!).*

We proceed similarly to the proof of Theorem 1.12.

*Proof.* Again, the first condition (by Proposition 1.6) implies that $\mathfrak{CC}(G;\mathcal{H})$ has diameter at most $R_0$; for the second, we consider a loop $L$ of length at most $2R_0+1$ in $\mathfrak{CC}(G;\mathcal{H})$. By Corollary 3.32 there exists a word $w$ over $\bigcup \mathcal{H}$ of length at most $2R_0+1$ such that $\triangle_\pi(L) \leq (t+4\ell) \cdot \operatorname{area}(w; \mathcal{R}_{3,0}(\triangle_\pi))$. By assumption and using Fact 3.21, $\operatorname{area}(w; \mathcal{R}_3^{\text{common}} \triangle_\pi) \leq \operatorname{area}(w; \mathcal{R}_{3,0}(\triangle_\pi)) \leq \delta$. Hence $\triangle_\pi(L) \leq (t+4\ell) \cdot \delta = O(\delta(t+\ell))$. Therefore, $\mathfrak{CC}(G;\mathcal{H})$ is $(2R_0+1, O((t+\ell)\delta))$-homotopically taut, and we can conclude using Theorem C.18. $\square$

# D Explicit definitions of and statements for unipotent groups

In this appendix, we explicitly define the groups in and formally state our Theorem 5.5.

### D.1 Explicit definitions of ungraded groups

**Definition D.1** (Explicit definition of $U_{A_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $U_{A_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $\{\{\zeta, t\}\}$ for $\zeta \in \Phi^+_{A_3}$ and $t \in \mathbb{F}_q$. The relations are, for every $t, u \in \mathbb{F}_q$, the commutator relations:

$$
\begin{aligned}
&[\{\{\alpha,t\}\},\{\{\beta,u\}\}] = \{\{\alpha+\beta,tu\}\},\\
&[\{\{\alpha,t\}\},\{\{\alpha+\beta,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\alpha+\beta,u\}\}] = \mathbb{1},\\
&[\{\{\alpha,t\}\},\{\{\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\gamma,u\}\}] = \{\{\beta+\gamma,tu\}\},\\
&[\{\{\beta,t\}\},\{\{\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\gamma,t\}\},\{\{\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta,t\}\},\{\{\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\alpha,t\}\},\{\{\beta+\gamma,u\}\}] = \{\{\alpha+\beta+\gamma,tu\}\},\\
&[\{\{\alpha+\beta,t\}\},\{\{\gamma,u\}\}] = \{\{\alpha+\beta+\gamma,tu\}\},\\
&[\{\{\alpha,t\}\},\{\{\alpha+\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\alpha+\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\gamma,t\}\},\{\{\alpha+\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta,t\}\},\{\{\alpha+\beta+\gamma,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\gamma,t\}\},\{\{\alpha+\beta+\gamma,u\}\}] = \mathbb{1}.
\end{aligned}
$$

and, for every $\zeta \in \Phi^+_{A_3}$, the linearity relations:

$$\{\{\zeta,t\}\} \cdot \{\{\zeta,u\}\} = \{\{\zeta,t+u\}\}.$$

$\lozenge$

**Remark D.2.** One can check that the following matrix realizations of these elements satisfy the above relations:

$$\{\{\alpha,t\}\} \mapsto \begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \{\{\beta,t\}\} \mapsto \begin{pmatrix} 1 & & & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \{\{\gamma,t\}\} \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & t \\ & & & 1 \end{pmatrix},$$

$$\{\{\alpha+\beta,t\}\} \mapsto \begin{pmatrix} 1 & & t & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \{\{\beta+\gamma,t\}\} \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & t \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \{\{\alpha+\beta+\gamma,t\}\} \mapsto \begin{pmatrix} 1 & & & t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

$\Diamond$

**Remark D.3.** One can also check that $\{\{\alpha,\cdot\}\}$ and $\{\{\beta,\cdot\}\}$ elements generate a subgroup wherein every element has a unique form $\{\{\alpha,\cdot\}\}\{\{\beta,\cdot\}\}\{\{\alpha+\beta,\cdot\}\}$, and similarly for $\{\{\alpha,\cdot\}\}$ and $\{\{\gamma,\cdot\}\}$ elements (all elements have the form $\{\{\alpha,\cdot\}\}\{\{\gamma,\cdot\}\}$) and for $\{\{\beta,\cdot\}\}$ and $\{\{\gamma,\cdot\}\}$ elements (all elements have the form $\{\{\beta,\cdot\}\}\{\{\gamma,\cdot\}\}\{\{\beta+\gamma,\cdot\}\}$). $\Diamond$

We give similar definitions for the $B_3$ unipotent groups:

**Definition D.4** (Explicit definition of $U^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $U^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $\{\{\zeta,t\}\}$ for $\zeta \in \Phi^{\mathrm{sm},+}_{B_3}$ and $t \in \mathbb{F}_q$. The relations are, for every $t,u \in \mathbb{F}_q$, the commutator relations:

$$\begin{aligned}
&[\{\{\beta,t\}\},\{\{\psi,u\}\}] = \{\{\beta+\psi,tu\}\}\{\{\beta+2\psi,tu^2\}\},\\
&[\{\{\beta,t\}\},\{\{\beta+\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\psi,t\}\},\{\{\beta+\psi,u\}\}] = \{\{\beta+2\psi,2tu\}\},\\
&[\{\{\psi,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\omega,u\}\}] = \mathbb{1},\\
&[\{\{\psi,t\}\},\{\{\omega,u\}\}] = \{\{\psi+\omega,2tu\}\},\\
&[\{\{\psi,t\}\},\{\{\psi+\omega,u\}\}] = \mathbb{1},\\
&[\{\{\omega,t\}\},\{\{\psi+\omega,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\psi+\omega,u\}\}] = \mathbb{1},\\
&[\{\{\beta+2\psi,t\}\},\{\{\omega,u\}\}] = \mathbb{1},\\
&[\{\{\beta+2\psi,t\}\},\{\{\psi+\omega,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\omega,u\}\}] = \{\{\beta+\psi+\omega,2tu\}\},\\
&[\{\{\beta,t\}\},\{\{\psi+\omega,u\}\}] = \{\{\beta+\psi+\omega,tu\}\},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\omega,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\beta,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\psi+\omega,t\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\beta+\psi,t\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi+\omega,t\}\},\{\{\beta+2\psi,t\}\}] = \mathbb{1},
\end{aligned}$$

and, for every $\zeta \in \Phi^{\mathrm{sm},+}_{B_3}$, the linearity relations:

$$\{\{\zeta,t\}\} \cdot \{\{\zeta,u\}\} = \{\{\zeta,t+u\}\}.$$

$\diamondsuit$

**Remark D.5.** One can check that the following matrix realizations of these elements satisfy the above relations:

$$\{\{\beta,t\}\} \mapsto \begin{pmatrix} 1 & -t & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & t \\ & & & & & & 1 \end{pmatrix}, \qquad \{\{\psi,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ -t & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ -t^2 & & & 2t & & & 1 \end{pmatrix},$$

$$\{\{\omega,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & -2t & -t^2 & & \\ & & & 1 & t & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}, \qquad \{\{\beta+\psi,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & -t & & 1 & & & \\ & & & & 1 & & \\ & -t^2 & & 2t & & 1 & \\ & & & & & & 1 \end{pmatrix},$$

$$\{\{\psi+\omega,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ -t & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & t & & 1 \end{pmatrix}, \qquad \{\{\beta+2\psi,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ t & & & & & 1 & \\ & -t & & & & & 1 \end{pmatrix},$$

$$\{\{\beta+\psi+\omega,t\}\} \mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & -t & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & t & 1 & \\ & & & & & & 1 \end{pmatrix}.$$

$\diamondsuit$

**Definition D.6** (Explicit definition of $U^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $U^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $\{\{\zeta,t\}\}$ for $\zeta \in \Phi^{\mathrm{lg},+}_{B_3}$ and $t \in \mathbb{F}_q$. The relations are, for every $t,u \in \mathbb{F}_q$, the commutator relations:

$$\begin{aligned}
&[\{\{\alpha,t\}\},\{\{\beta,u\}\}] = \{\{\alpha+\beta,tu\}\},\\
&[\{\{\alpha,t\}\},\{\{\alpha+\beta,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\alpha+\beta,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\psi,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\beta,t\}\},\{\{\psi,u\}\}] = \{\{\beta+\psi,tu\}\}\{\{\beta+2\psi,tu^2\}\},\\
&[\{\{\beta,t\}\},\{\{\beta+\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\psi,t\}\},\{\{\beta+\psi,u\}\}] = \{\{\beta+2\psi,2tu\}\},\\
&[\{\{\psi,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\beta+2\psi,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\alpha+\beta,t\}\},\{\{\beta+\psi,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\alpha+\beta+\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\alpha+\beta+\psi,u\}\}] = \mathbb{1},
\end{aligned}$$

$$
\begin{aligned}
&[\{\{\alpha+\beta,t\}\},\{\{\alpha+\beta+\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+2\psi,t\}\},\{\{\alpha+\beta+\psi,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\beta+2\psi,u\}\}] = \{\{\alpha+\beta+2\psi,tu\}\},\\
&[\{\{\alpha+\beta+\psi,u\}\},\{\{\psi,t\}\}] = \{\{\alpha+\beta+2\psi,-2tu\}\},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\psi,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+2\psi,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta+\psi,t\}\},\{\{\alpha+\beta+2\psi,u\}\}] = \mathbb{1},\\[4pt]
&[\{\{\alpha+\beta,t\}\},\{\{\psi,u\}\}] = \{\{\alpha+\beta+\psi,tu\}\}\{\{\alpha+\beta+2\psi,tu^2\}\},\\[4pt]
&[\{\{\alpha+\beta,t\}\},\{\{\beta+2\psi,u\}\}] = \{\{\alpha+2\beta+2\psi,-tu\}\},\\
&[\{\{\alpha+\beta+\psi,t\}\},\{\{\beta+\psi,u\}\}] = \{\{\alpha+2\beta+2\psi,-2tu\}\},\\
&[\{\{\alpha+\beta+2\psi,t\}\},\{\{\beta,u\}\}] = \{\{\alpha+2\beta+2\psi,-tu\}\},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\beta+\psi,u\}\}] = \{\{\alpha+\beta+\psi,tu\}\}\{\{\alpha+2\beta+2\psi,tu^2\}\},\\[4pt]
&[\{\{\alpha,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\psi,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+\psi,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\beta+2\psi,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta+\psi,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},\\
&[\{\{\alpha+\beta+2\psi,t\}\},\{\{\alpha+2\beta+2\psi,u\}\}] = \mathbb{1},
\end{aligned}
$$

and, for every $\zeta \in \Phi_{B_3}^{\mathrm{lg},+}$, the linearity relation and homogeneous commutator relations:

$$
\begin{aligned}
&\{\{\alpha+\beta+\psi,t\}\}\{\{\alpha+\beta+\psi,u\}\} = \{\{\alpha+\beta+\psi,t+u\}\},\\
&[\{\{\alpha+\beta+\psi,t\}\},\{\{\alpha+\beta+\psi,u\}\}] = \mathbb{1}.
\end{aligned}
$$

$\Diamond$

**Remark D.7.** One can check that the following matrix realizations of these elements satisfy the above relations:

$$
\{\{\alpha,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 & -t &&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 & t & \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}, \quad \{\{\beta,t\}\} \mapsto \begin{pmatrix} 1 & -t &&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ &&&&& 1 & t \\ &&&&&& 1 \end{pmatrix},
$$

$$
\{\{\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ -t &&& 1 &&& \\ &&&& 1 && \\ &&&&& 1 & \\ -t^2 &&& 2t &&& 1 \end{pmatrix}, \quad \{\{\alpha+\beta,t\}\} \mapsto \begin{pmatrix} 1 && -t &&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && t \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix},
$$

$$\{\{\beta+\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ & -t && 1 &&& \\ &&&& 1 && \\ & -t^2 && 2t && 1 & \\ &&&&&& 1 \end{pmatrix}, \quad \{\{\alpha+\beta+\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ && -t & 1 &&& \\ && -t^2 & 2t & 1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix},$$

$$\{\{\beta+2\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ &&&& 1 && \\ t &&&&& 1 & \\ & -t &&&&& 1 \end{pmatrix}, \quad \{\{\alpha+\beta+2\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ t &&&& 1 && \\ &&&&& 1 & \\ && -t &&&& 1 \end{pmatrix},$$

$$\{\{\alpha+2\beta+2\psi,t\}\} \mapsto \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&& 1 &&& \\ & t &&& 1 && \\ && -t &&& 1 & \\ &&&&&& 1 \end{pmatrix}.$$

$\Diamond$

## D.2 Explicit definitions of graded groups

Recall our convention that $[n] = \{0,\dots,n\}$.

**Definition D.8** (Explicit definition of $GU_{A_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $GU_{A_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $(\!(\zeta,t,i)\!)$ for $\zeta \in \Phi^+_{A_3}$, $t \in \mathbb{F}_q$, and $i \in [\mathrm{height}_{A_3}(\zeta)]$. The relations are, for every $t,u \in \mathbb{F}_q$, the heterogeneous commutator relations:

$$\begin{aligned}
&\forall i,j \in [1], && [(\!(\alpha,t,i)\!),(\!(\beta,u,j)\!)] = (\!(\alpha+\beta,tu,i+j)\!), \\
&\forall i \in [1], j \in [2], && [(\!(\alpha,t,i)\!),(\!(\alpha+\beta,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [1], j \in [2], && [(\!(\beta,t,i)\!),(\!(\alpha+\beta,u,j)\!)] = \mathbb{1}, \\
&\forall i,j \in [1], && [(\!(\alpha,t,i)\!),(\!(\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i,j \in [1], && [(\!(\beta,t,i)\!),(\!(\gamma,u,j)\!)] = (\!(\beta+\gamma,tu,i+j)\!), \\
&\forall i \in [1], j \in [2], && [(\!(\beta,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [1], j \in [2], && [(\!(\gamma,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i,j \in [2], && [(\!(\alpha+\beta,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [1], j \in [2], && [(\!(\alpha,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = (\!(\alpha+\beta+\gamma,tu,i+j)\!), \\
&\forall i \in [2], j \in [1], && [(\!(\alpha+\beta,t,i)\!),(\!(\gamma,u,j)\!)] = (\!(\alpha+\beta+\gamma,tu,i+j)\!), \\
&\forall i \in [1], j \in [3], && [(\!(\alpha,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [1], j \in [3], && [(\!(\beta,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [1], j \in [3], && [(\!(\gamma,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [2], j \in [3], && [(\!(\alpha+\beta,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1}, \\
&\forall i \in [2], j \in [3], && [(\!(\beta+\gamma,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},
\end{aligned}$$

and, for every $\zeta \in \Phi^+_{A_3}$, the linearity and homogeneous commutator relations:

$$\forall i \in [\mathrm{height}_{A_3}(\zeta)], \qquad (\!(\zeta,t,i)\!)(\!(\zeta,u,i)\!) = (\!(\zeta,t+u,i+j)\!),$$

$$\forall i,j \in [\mathrm{height}_{A_3}(\zeta)], \qquad [(\!(\zeta,t,i)\!),(\!(\zeta,u,j)\!)] = \mathbb{1}.$$

$\Diamond$

**Remark D.9.** The lifting homomorphism defined in Theorem 5.1 is easy to instantiate concretely once the corresponding ungraded and graded groups have been defined. For instance, now that we have the definitions of $U_{A_3}(\mathbb{F}_q)$ and $GU_{A_3}(\mathbb{F}_{q^k})$, given values $t_1,t_0,u_1,u_0,v_1,v_0 \in \mathbb{F}_{q^k}$, the homomorphism is defined via:

$$\begin{aligned}
\{\{\alpha,w\}\} &\mapsto (\!(\alpha,wt_1,1)\!)(\!(\alpha,wt_0,0)\!),\\
\{\{\beta,w\}\} &\mapsto (\!(\beta,wu_1,1)\!)(\!(\beta,wu_0,0)\!),\\
\{\{\gamma,w\}\} &\mapsto (\!(\gamma,wv_1,1)\!)(\!(\gamma,wv_0,0)\!),\\
\{\{\alpha+\beta,w\}\} &\mapsto (\!(\alpha+\beta,wt_1u_1,2)\!)(\!(\alpha+\beta,w(t_1u_0+t_0u_1),1)\!)(\!(\alpha+\beta,wt_0u_0,0)\!),\\
\{\{\beta+\gamma,w\}\} &\mapsto (\!(\beta+\gamma,wu_1v_1,2)\!)(\!(\beta+\gamma,w(u_1v_0+u_0v_1),1)\!)(\!(\beta+\gamma,wu_0v_0,0)\!),\\
\{\{\alpha+\beta+\gamma,w\}\} &\mapsto (\!(\alpha+\beta+\gamma,wt_1u_1v_1,3)\!)(\!(\alpha+\beta+\gamma,w(t_1u_1v_0+t_1u_0v_1+t_0u_1v_1),2)\!)\\
&\qquad \cdot (\!(\alpha+\beta+\gamma,w(t_1u_0v_0+t_0u_1v_0+t_0u_0v_1),1)\!)(\!(\alpha+\beta+\gamma,wt_0u_0v_0,0)\!).
\end{aligned}$$

$\Diamond$

**Definition D.10** (Explicit definition of $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $(\!(\zeta,t,i)\!)$ for $\zeta \in \Phi^{\mathrm{sm},+}_{B_3}$, $t \in \mathbb{F}_q$, and $i \in [\mathrm{height}^{\mathrm{sm}}_{B_3}(\zeta)]$. The relations are, for every $t,u \in \mathbb{F}_q$, the heterogeneous commutator relations:

$$\begin{aligned}
\forall i,j \in [1], &\qquad [(\!(\beta,t,i)\!),(\!(\psi,u,j)\!)] = (\!(\beta+\psi,tu,i+j)\!)(\!(\beta+2\psi,tu^2,i+2j)\!),\\
\forall i \in [1], j \in [2], &\qquad [(\!(\beta,t,i)\!),(\!(\beta+\psi,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [3], &\qquad [(\!(\beta,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [2], &\qquad [(\!(\psi,t,i)\!),(\!(\beta+\psi,u,j)\!)] = (\!(\beta+2\psi,2tu,i+j)\!),\\
\forall i \in [1], j \in [3], &\qquad [(\!(\psi,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},\\
\forall i \in [2], j \in [3], &\qquad [(\!(\beta+\psi,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},
\end{aligned}$$

$$\forall i,j \in [1], \qquad [(\!(\beta,t,i)\!),(\!(\omega,u,j)\!)] = \mathbb{1},$$

$$\begin{aligned}
\forall i,j \in [1], &\qquad [(\!(\psi,t,i)\!),(\!(\omega,u,j)\!)] = (\!(\psi+\omega,2tu,i+j)\!),\\
\forall i \in [1], j \in [2], &\qquad [(\!(\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [2], &\qquad [(\!(\omega,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},
\end{aligned}$$

$$\begin{aligned}
\forall i,j \in [2], &\qquad [(\!(\beta+\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [1], &\qquad [(\!(\beta+2\psi,t,i)\!),(\!(\omega,u,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [2], &\qquad [(\!(\beta+2\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},
\end{aligned}$$

$$\begin{aligned}
\forall i \in [2], j \in [1], &\qquad [(\!(\beta+\psi,t,i)\!),(\!(\omega,u,j)\!)] = (\!(\beta+\psi+\omega,2tu,i+j)\!),\\
\forall i \in [1], j \in [2], &\qquad [(\!(\beta,t,i)\!),(\!(\psi+\omega,u,j)\!)] = (\!(\beta+\psi+\omega,tu,i+j)\!),
\end{aligned}$$

$$\begin{aligned}
\forall i \in [3], j \in [1], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\omega,u,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [1], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\beta,u,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [1], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\psi,u,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [2], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\psi+\omega,t,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [2], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\beta+\psi,t,j)\!)] = \mathbb{1},\\
\forall i \in [3], j \in [3], &\qquad [(\!(\beta+\psi+\omega,t,i)\!),(\!(\beta+2\psi,t,j)\!)] = \mathbb{1},
\end{aligned}$$

and, for every $\zeta \in \Phi^{\mathrm{sm},+}_{B_3}$, the linearity and homogeneous commutator relations:

$$\begin{aligned}
\forall i \in [\mathrm{height}^{\mathrm{sm}}_{B_3}(\zeta)], &\qquad (\!(\zeta,t,i)\!)(\!(\zeta,u,i)\!) = (\!(\zeta,t+u,i+j)\!),\\
\forall i,j \in [\mathrm{height}^{\mathrm{sm}}_{B_3}(\zeta)], &\qquad [(\!(\zeta,t,i)\!),(\!(\zeta,u,j)\!)] = \mathbb{1}.
\end{aligned}$$

$\Diamond$

**Definition D.11** (Explicit definition of $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$)**.** Let $q$ be a prime power. The group $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$ is given by the following presentation: The generators are symbols $(\!(\zeta, t, i)\!)$ for $\zeta \in \Phi^{\mathrm{lg},+}_{B_3}$, $t \in \mathbb{F}_q$, and $i \in [\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)]$. The relations are, for every $t, u \in \mathbb{F}_q$, the heterogeneous commutator relations:

$$\begin{aligned}
\forall i, j \in [1], &\quad [(\!(\alpha, t, i)\!), (\!(\beta, u, j)\!)] = (\!(\alpha + \beta, tu, i + j)\!), \\
\forall i \in [1], j \in [2], &\quad [(\!(\alpha, t, i)\!), (\!(\alpha + \beta, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [2], &\quad [(\!(\beta, t, i)\!), (\!(\alpha + \beta, u, j)\!)] = \mathbb{1}, \\
\\
\forall i \in [1], j \in [1], &\quad [(\!(\alpha, t, i)\!), (\!(\psi, u, j)\!)] = \mathbb{1}, \\
\\
\forall i, j \in [1], &\quad [(\!(\beta, t, i)\!), (\!(\psi, u, j)\!)] = (\!(\beta + \psi, tu, i + j)\!)(\!(\beta + 2\psi, tu^2, i + 2j)\!), \\
\forall i \in [1], j \in [2], &\quad [(\!(\beta, t, i)\!), (\!(\beta + \psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [3], &\quad [(\!(\beta, t, i)\!), (\!(\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [2], &\quad [(\!(\psi, t, i)\!), (\!(\beta + \psi, u, j)\!)] = (\!(\beta + 2\psi, 2tu, i + j)\!), \\
\forall i \in [1], j \in [3], &\quad [(\!(\psi, t, i)\!), (\!(\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [3], &\quad [(\!(\beta + \psi, t, i)\!), (\!(\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\\
\forall i, j \in [2], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\beta + \psi, u, j)\!)] = \mathbb{1}, \\
\\
\forall i \in [1], j \in [3], &\quad [(\!(\alpha, t, i)\!), (\!(\alpha + \beta + \psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [3], &\quad [(\!(\beta, t, i)\!), (\!(\alpha + \beta + \psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [3], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\alpha + \beta + \psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [3], j \in [3], &\quad [(\!(\beta + 2\psi, t, i)\!), (\!(\alpha + \beta + \psi, u, j)\!)] = \mathbb{1}, \\
\\
\forall i \in [1], j \in [3], &\quad [(\!(\alpha, t, i)\!), (\!(\beta + 2\psi, u, j)\!)] = (\!(\alpha + \beta + 2\psi, tu, i + j)\!), \\
\forall i \in [1], j \in [3], &\quad [(\!(\alpha + \beta + \psi, u, i)\!), (\!(\psi, t, j)\!)] = (\!(\alpha + \beta + 2\psi, -2tu, i + j)\!), \\
\\
\forall i \in [1], j \in [4], &\quad [(\!(\alpha, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [4], &\quad [(\!(\psi, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [4], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [4], &\quad [(\!(\beta + \psi, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [3], j \in [4], &\quad [(\!(\beta + 2\psi, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [3], j \in [4], &\quad [(\!(\alpha + \beta + \psi, t, i)\!), (\!(\alpha + \beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\\
\forall i \in [2], j \in [1], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\psi, u, j)\!)] = (\!(\alpha + \beta + \psi, tu, i + j)\!)(\!(\alpha + \beta + 2\psi, tu^2, i + 2j)\!), \\
\\
\forall i \in [2], j \in [3], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\beta + 2\psi, u, j)\!)] = (\!(\alpha + 2\beta + 2\psi, -tu, i + j)\!), \\
\forall i \in [3], j \in [2], &\quad [(\!(\alpha + \beta + \psi, t, i)\!), (\!(\beta + \psi, u, j)\!)] = (\!(\alpha + 2\beta + 2\psi, -2tu, i + j)\!), \\
\forall i \in [4], j \in [1], &\quad [(\!(\alpha + \beta + 2\psi, t, i)\!), (\!(\beta, u, j)\!)] = (\!(\alpha + 2\beta + 2\psi, -tu, i + j)\!), \\
\\
\forall i \in [1], j \in [2], &\quad [(\!(\alpha, t, i)\!), (\!(\beta + \psi, u, j)\!)] = (\!(\alpha + \beta + \psi, tu, i + j)\!)(\!(\alpha + 2\beta + 2\psi, tu^2, i + 2j)\!), \\
\\
\forall i \in [1], j \in [5], &\quad [(\!(\alpha, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [5], &\quad [(\!(\beta, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [5], &\quad [(\!(\psi, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [5], &\quad [(\!(\alpha + \beta, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [5], &\quad [(\!(\beta + \psi, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [3], j \in [5], &\quad [(\!(\beta + 2\psi, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [3], j \in [5], &\quad [(\!(\alpha + \beta + \psi, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [4], j \in [5], &\quad [(\!(\alpha + \beta + 2\psi, t, i)\!), (\!(\alpha + 2\beta + 2\psi, u, j)\!)] = \mathbb{1},
\end{aligned}$$

and, for every $\zeta \in \Phi^{\mathrm{lg},+}_{B_3}$, the linearity relation and homogeneous commutator relations:

$$\begin{aligned}
\forall i \in [\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)], &\qquad (\!(\alpha+\beta+\psi,t,i)\!)(\!(\alpha+\beta+\psi,u,i)\!) = (\!(\alpha+\beta+\psi,t+u,i)\!),\\
\forall i,j \in [\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)], &\qquad [(\!(\alpha+\beta+\psi,t,i)\!),(\!(\alpha+\beta+\psi,u,j)\!)] = \mathbb{1}.
\end{aligned}$$

$\Diamond$

## D.3 Explicit statements of lifting theorems

**Theorem D.12** (Lifting theorem for $GU_{A_3}(\mathbb{F}_q)$ [KO21])**.** *There exist absolute constants $r_0$ and $\delta$ s.t. the following holds. Let $p$ be a prime (power) and $k \geq 1 \in \mathbb{N}$. For every $t \in \mathbb{F}_{p^k}$ and $i \in [3]$, there exists an alias word $\langle\!\langle \alpha+\beta+\gamma,t,i\rangle\!\rangle$, of length at most $r_0$, over the designated subgroups of $GU_{A_3}(\mathbb{F}_q)$ which evaluates to $(\!(\alpha+\beta+\gamma,t,i)\!)$, s.t. the following holds. Consider the following set of relations over the designated subgroups of $GU_{A_3}(\mathbb{F}_q)$: For every $t,u \in \mathbb{F}_{p^k}$,*

$$\begin{aligned}
\forall i,j \in [1], &\qquad [(\!(\alpha,t,i)\!),(\!(\beta,u,j)\!)] = (\!(\alpha+\beta,tu,i+j)\!),\\
\forall i \in [1], j \in [2], &\qquad [(\!(\alpha,t,i)\!),(\!(\alpha+\beta,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [2], &\qquad [(\!(\beta,t,i)\!),(\!(\alpha+\beta,u,j)\!)] = \mathbb{1},\\
\forall i,j \in [1], &\qquad [(\!(\alpha,t,i)\!),(\!(\gamma,u,j)\!)] = \mathbb{1},\\
\forall i,j \in [1], &\qquad [(\!(\beta,t,i)\!),(\!(\gamma,u,j)\!)] = (\!(\beta+\gamma,tu,i+j)\!),\\
\forall i \in [1], j \in [2], &\qquad [(\!(\beta,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [2], &\qquad [(\!(\gamma,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1},
\end{aligned}$$

*and for every $\zeta \in \Phi^+_{A_3} \setminus \{\alpha+\beta+\gamma\}$,*

$$\begin{aligned}
\forall i \in [\mathrm{height}(\zeta)], &\qquad (\!(\zeta,t,i)\!)(\!(\zeta,u,i)\!) = (\!(\zeta,t+u,i)\!),\\
\forall i,j \in [\mathrm{height}(\zeta)], &\qquad [(\!(\zeta,t,i)\!),(\!(\zeta,u,j)\!)] = \mathbb{1},
\end{aligned}$$

*as well as all relators which are lifts (in the sense of Theorem 5.1) of relators in $U_{A_3}(\mathbb{F}_p)$ of length at most $r_0$. From the aforementioned relators, for every $t,u \in \mathbb{F}_{p^k}$, it is possible derive in at most $\delta$ steps: The heterogeneous commutator relations:*

$$\begin{aligned}
\forall i,j \in [2], &\qquad [(\!(\alpha+\beta,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [2], &\qquad [(\!(\alpha,t,i)\!),(\!(\beta+\gamma,u,j)\!)] = (\!(\alpha+\beta+\gamma,tu,i+j)\!),\\
\forall i \in [2], j \in [1], &\qquad [(\!(\alpha+\beta,t,i)\!),(\!(\gamma,u,j)\!)] = (\!(\alpha+\beta+\gamma,tu,i+j)\!),\\
\forall i \in [1], j \in [3], &\qquad [(\!(\alpha,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [3], &\qquad [(\!(\beta,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [1], j \in [3], &\qquad [(\!(\gamma,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [2], j \in [3], &\qquad [(\!(\alpha+\beta,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},\\
\forall i \in [2], j \in [3], &\qquad [(\!(\beta+\gamma,t,i)\!),(\!(\alpha+\beta+\gamma,u,j)\!)] = \mathbb{1},
\end{aligned}$$

*and the linearity and homogeneous commutator relations:*

$$\begin{aligned}
\forall i \in [3], &\qquad \langle\!\langle \alpha+\beta+\gamma,t,i\rangle\!\rangle\langle\!\langle \alpha+\beta+\gamma,u,i\rangle\!\rangle = \langle\!\langle \alpha+\beta+\gamma,t+u,i+j\rangle\!\rangle,\\
\forall i,j \in [3], &\qquad [\langle\!\langle \alpha+\beta+\gamma,t,i\rangle\!\rangle,\langle\!\langle \alpha+\beta+\gamma,u,j\rangle\!\rangle] = \mathbb{1}.
\end{aligned}$$

**Theorem D.13** (Lifting theorem for $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$, formally verified in [WBCS25])**.** *There exist absolute constants $r_0$ and $\delta$ s.t. the following holds. Let $p$ be a prime (power) and $k \geq 1 \in \mathbb{N}$. For every $t \in \mathbb{F}_{p^k}$ and $i \in [3]$, there exists an alias word $\langle\!\langle \beta+\psi+\omega,t,i\rangle\!\rangle$, of length at most $r_0$, over the designated subgroups* *of $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$ which evaluates to $(\!(\beta+\psi+\omega,t,i)\!)$, s.t. the following holds. Consider the following set of relations over the designated subgroups of $GU^{\mathrm{sm}}_{B_3}(\mathbb{F}_q)$: For every $t,u\in\mathbb{F}_{p^k}$,*

$$\begin{aligned}
&\forall i,j\in[1], && [(\!(\beta,t,i)\!),(\!(\psi,u,j)\!)] = (\!(\beta+\psi,tu,i+j)\!)(\!(\beta+2\psi,tu^2,i+2j)\!),\\
&\forall i\in[1],j\in[2], && [(\!(\beta,t,i)\!),(\!(\beta+\psi,u,j)\!)] = \mathbb{1},\\
&\forall i\in[1],j\in[3], && [(\!(\beta,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},\\
&\forall i\in[1],j\in[2], && [(\!(\psi,t,i)\!),(\!(\beta+\psi,u,j)\!)] = (\!(\beta+2\psi,2tu,i+j)\!),\\
&\forall i\in[1],j\in[3], && [(\!(\psi,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},\\
&\forall i\in[2],j\in[3], && [(\!(\beta+\psi,t,i)\!),(\!(\beta+2\psi,u,j)\!)] = \mathbb{1},
\end{aligned}$$

$$\forall i,j\in[1],\qquad [(\!(\beta,t,i)\!),(\!(\omega,u,j)\!)] = \mathbb{1},$$

$$\begin{aligned}
&\forall i,j\in[1], && [(\!(\psi,t,i)\!),(\!(\omega,u,j)\!)] = (\!(\psi+\omega,2tu,i+j)\!),\\
&\forall i\in[1],j\in[2], && [(\!(\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},\\
&\forall i\in[1],j\in[2], && [(\!(\omega,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},
\end{aligned}$$

*and for every $\zeta\in\Phi^{\mathrm{sm},+}_{B_3}\setminus\{\beta+\psi+\omega\}$,*

$$\begin{aligned}
&\forall i\in[\mathrm{height}(\zeta)], && (\!(\zeta,t,i)\!)(\!(\zeta,u,i)\!) = (\!(\zeta,t+u,i)\!),\\
&\forall i,j\in[\mathrm{height}(\zeta)], && [(\!(\zeta,t,i)\!),(\!(\zeta,u,j)\!)] = \mathbb{1},
\end{aligned}$$

*as well as all relators which are lifts (in the sense of Theorem 5.1) of relators in $U^{\mathrm{sm}}_{B_3}(\mathbb{F}_p)$ of length at most $r_0$. From the aforementioned relators, for every $t,u\in\mathbb{F}_{p^k}$, it is possible derive in at most $\delta$ steps: The heterogeneous commutator relations:*

$$\begin{aligned}
&\forall i,j\in[2], && [(\!(\beta+\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[1], && [(\!(\beta+2\psi,t,i)\!),(\!(\omega,u,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[2], && [(\!(\beta+2\psi,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \mathbb{1},
\end{aligned}$$

$$\begin{aligned}
&\forall i\in[2],j\in[1], && [(\!(\beta+\psi,t,i)\!),(\!(\omega,u,j)\!)] = \langle\!\langle\beta+\psi+\omega,2tu,i+j\rangle\!\rangle,\\
&\forall i\in[1],j\in[2], && [(\!(\beta,t,i)\!),(\!(\psi+\omega,u,j)\!)] = \langle\!\langle\beta+\psi+\omega,tu,i+j\rangle\!\rangle,
\end{aligned}$$

$$\begin{aligned}
&\forall i\in[3],j\in[1], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\omega,u,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[1], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\beta,u,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[1], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\psi,u,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[2], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\psi+\omega,t,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[2], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\beta+\psi,t,j)\!)] = \mathbb{1},\\
&\forall i\in[3],j\in[3], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,(\!(\beta+2\psi,t,j)\!)] = \mathbb{1},
\end{aligned}$$

*and the linearity and homogeneous commutator relations:*

$$\begin{aligned}
&\forall i\in[3], && \langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle\langle\!\langle\beta+\psi+\omega,u,i\rangle\!\rangle = \langle\!\langle\beta+\psi+\omega,t+u,i+j\rangle\!\rangle,\\
&\forall i,j\in[3], && [\langle\!\langle\beta+\psi+\omega,t,i\rangle\!\rangle,\langle\!\langle\beta+\psi+\omega,u,j\rangle\!\rangle] = \mathbb{1}.
\end{aligned}$$

**Theorem D.14** (Lifting theorem for $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$, formally verified in [WBCS25])**.** *There exist absolute constants $r_0$ and $\delta$ s.t. the following holds. Let $p$ be a prime (power) and $k\geq 1\in\mathbb{N}$. For every $t\in\mathbb{F}_{p^k}$, $\zeta\in\{\alpha+\beta+\psi,\alpha+\beta+2\psi,\alpha+2\beta+2\psi\}$ and $i\in[\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)]$, there exists an alias word $\langle\!\langle\zeta,t,i\rangle\!\rangle$, of length at most $r_0$, over the designated subgroups of $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$ which evaluates to $(\!(\zeta,t,i)\!)$, s.t. the following holds. Consider the following set of relations over the designated subgroups of $GU^{\mathrm{lg}}_{B_3}(\mathbb{F}_q)$: For every $t,u\in\mathbb{F}_{p^k}$,*

$$\forall i,j\in[1],\qquad [(\!(\alpha,t,i)\!),(\!(\beta,u,j)\!)] = (\!(\alpha+\beta,tu,i+j)\!),$$

$$
\begin{aligned}
\forall i \in [1], j \in [2], \quad & [(\!(\alpha, t, i)\!), (\!(\alpha+\beta, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [2], \quad & [(\!(\beta, t, i)\!), (\!(\alpha+\beta, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [1], \quad & [(\!(\alpha, t, i)\!), (\!(\psi, u, j)\!)] = \mathbb{1}, \\
\forall i, j \in [1], \quad & [(\!(\beta, t, i)\!), (\!(\psi, u, j)\!)] = (\!(\beta+\psi, tu, i+j)\!)(\!(\beta+2\psi, tu^2, i+2j)\!), \\
\forall i \in [1], j \in [2], \quad & [(\!(\beta, t, i)\!), (\!(\beta+\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [3], \quad & [(\!(\beta, t, i)\!), (\!(\beta+2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [2], \quad & [(\!(\psi, t, i)\!), (\!(\beta+\psi, u, j)\!)] = (\!(\beta+2\psi, 2tu, i+j)\!), \\
\forall i \in [1], j \in [3], \quad & [(\!(\psi, t, i)\!), (\!(\beta+2\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [2], j \in [3], \quad & [(\!(\beta+\psi, t, i)\!), (\!(\beta+2\psi, u, j)\!)] = \mathbb{1},
\end{aligned}
$$

*and for every* $\zeta \in \Phi_{B_3}^{\mathrm{lg},+} \setminus \{\alpha+\beta+\psi, \alpha+\beta+2\psi, \alpha+2\beta+2\psi\}$*, the linearity and homogeneous commutator relations*

$$
\begin{aligned}
\forall i \in [\mathrm{height}(\zeta)], \quad & (\!(\zeta, t, i)\!)(\!(\zeta, u, i)\!) = (\!(\zeta, t+u, i)\!), \\
\forall i, j \in [\mathrm{height}(\zeta)], \quad & [(\!(\zeta, t, i)\!), (\!(\zeta, u, j)\!)] = \mathbb{1},
\end{aligned}
$$

*as well as all relators which are lifts (in the sense of Theorem 5.1) of relators in* $U_{B_3}^{\mathrm{lg}}(\mathbb{F}_p)$ *of length at most* $r_0$*. From the aforementioned relators, for every* $t, u \in \mathbb{F}_{p^k}$*, it is possible derive in at most* $\delta$ *steps: The heterogeneous commutator relations:*

$$
\begin{aligned}
\forall i, j \in [2], \quad & [(\!(\alpha+\beta, t, i)\!), (\!(\beta+\psi, u, j)\!)] = \mathbb{1}, \\
\forall i \in [1], j \in [3], \quad & [(\!(\alpha, t, i)\!), \langle\!\langle \alpha+\beta+\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [1], j \in [3], \quad & [(\!(\beta, t, i)\!), \langle\!\langle \alpha+\beta+\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [3], \quad & [(\!(\alpha+\beta, t, i)\!), \langle\!\langle \alpha+\beta+\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [3], j \in [3], \quad & [(\!(\beta+2\psi, t, i)\!), \langle\!\langle \alpha+\beta+\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [1], j \in [3], \quad & [(\!(\alpha, t, i)\!), (\!(\beta+2\psi, u, j)\!)] = \langle\!\langle \alpha+\beta+2\psi, tu, i+j \rangle\!\rangle, \\
\forall i \in [1], j \in [3], \quad & [\langle\!\langle \alpha+\beta+\psi, u, i \rangle\!\rangle, (\!(\psi, t, j)\!)] = \langle\!\langle \alpha+\beta+2\psi, -2tu, i+j \rangle\!\rangle, \\
\forall i \in [1], j \in [4], \quad & [(\!(\alpha, t, i)\!), \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [1], j \in [4], \quad & [(\!(\psi, t, i)\!), \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [4], \quad & [(\!(\alpha+\beta, t, i)\!), \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [4], \quad & [(\!(\beta+\psi, t, i)\!), \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [3], j \in [4], \quad & [(\!(\beta+2\psi, t, i)\!), \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [3], j \in [4], \quad & [\langle\!\langle \alpha+\beta+\psi, t, i \rangle\!\rangle, \langle\!\langle \alpha+\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [1], \quad & [(\!(\alpha+\beta, t, i)\!), (\!(\psi, u, j)\!)] = \langle\!\langle \alpha+\beta+\psi, tu, i+j \rangle\!\rangle \langle\!\langle \alpha+\beta+2\psi, tu^2, i+2j \rangle\!\rangle, \\
\forall i \in [2], j \in [3], \quad & [(\!(\alpha+\beta, t, i)\!), (\!(\beta+2\psi, u, j)\!)] = \langle\!\langle \alpha+2\beta+2\psi, -tu, i+j \rangle\!\rangle, \\
\forall i \in [3], j \in [2], \quad & [\langle\!\langle \alpha+\beta+\psi, t, i \rangle\!\rangle, (\!(\beta+\psi, u, j)\!)] = \langle\!\langle \alpha+2\beta+2\psi, -2tu, i+j \rangle\!\rangle, \\
\forall i \in [4], j \in [1], \quad & [\langle\!\langle \alpha+\beta+2\psi, t, i \rangle\!\rangle, (\!(\beta, u, j)\!)] = \langle\!\langle \alpha+2\beta+2\psi, -tu, i+j \rangle\!\rangle, \\
\forall i \in [1], j \in [2], \quad & [(\!(\alpha, t, i)\!), (\!(\beta+\psi, u, j)\!)] = \langle\!\langle \alpha+\beta+\psi, tu, i+j \rangle\!\rangle \langle\!\langle \alpha+2\beta+2\psi, tu^2, i+2j \rangle\!\rangle, \\
\forall i \in [1], j \in [5], \quad & [(\!(\alpha, t, i)\!), \langle\!\langle \alpha+2\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [1], j \in [5], \quad & [(\!(\beta, t, i)\!), \langle\!\langle \alpha+2\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [1], j \in [5], \quad & [(\!(\psi, t, i)\!), \langle\!\langle \alpha+2\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [5], \quad & [(\!(\alpha+\beta, t, i)\!), \langle\!\langle \alpha+2\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1}, \\
\forall i \in [2], j \in [5], \quad & [(\!(\beta+\psi, t, i)\!), \langle\!\langle \alpha+2\beta+2\psi, u, j \rangle\!\rangle] = \mathbb{1},
\end{aligned}
$$

$$\forall i \in [3], j \in [5], \qquad [\langle\!\langle \beta + 2\psi, t, i \rangle\!\rangle, \langle\!\langle \alpha + 2\beta + 2\psi, u, j \rangle\!\rangle] = \mathbb{1},$$
$$\forall i \in [3], j \in [5], \qquad [\langle\!\langle \alpha + \beta + \psi, t, i \rangle\!\rangle, \langle\!\langle \alpha + 2\beta + 2\psi, u, j \rangle\!\rangle] = \mathbb{1},$$
$$\forall i \in [4], j \in [5], \qquad [\langle\!\langle \alpha + \beta + 2\psi, t, i \rangle\!\rangle, \langle\!\langle \alpha + 2\beta + 2\psi, u, j \rangle\!\rangle] = \mathbb{1},$$

*and, for every* $\zeta \in \{\alpha + \beta + \psi, \alpha + \beta + 2\psi, \alpha + 2\beta + 2\psi\}$*, the linearity relation and homogeneous commutator relations:*

$$\forall i \in [\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)], \qquad \langle\!\langle \zeta, t, i \rangle\!\rangle \langle\!\langle \zeta, u, i \rangle\!\rangle = \langle\!\langle \zeta, t + u, i \rangle\!\rangle,$$
$$\forall i, j \in [\mathrm{height}^{\mathrm{lg}}_{B_3}(\zeta)], \qquad [\langle\!\langle \zeta, t, i \rangle\!\rangle, \langle\!\langle \zeta, u, j \rangle\!\rangle] = \mathbb{1}.$$